Numerical Treatment of Singular Boundary and Initial Value Problems

Peter W. Kofler and Ewa B. Weinmüller
University of Technology Vienna
Institute of Applied Mathematics and Numerical Analysis
A-1040 Vienna, Austria

Abstract

In this paper we give an overview of the theory on the *singular* two-point BVPs for ODE's of second order. We discuss the solvability of the analytical problem and the stability and convergence results of a collocation method and a finite difference scheme when applied for its approximate solution. Moreover, experimental results of the performance of linear multistep methods in case of singular initial value problems will be presented and some open questions concerning the behavior of these methods will be addressed.

1 Overview of the Theory of Singular BVP's

Lasting interest in the numerical solution of singular boundary value problems is strongly motivated by numerous applications from chemistry, physics and mechanics, cf. [4] and [7]. During the last decades there was a considerable progress in the theory of approximations to solutions of singular problems for systems of ODE's of the first and second order. For wide classes of methods (finite difference methods, collocation, Runge–Kutta methods, shooting, parallel shooting, multistep methods) stability and convergence results have been established and global error bounds have been derived. Also, asymptotic expansions for the global error of the standard three-point finite difference scheme for a second order problem have been studied.

The second author's main contribution to the theory of singular problems was the investigation of the analytical properties of two-point nonlinear boundary value problems of second order and the study of the performance of several numerical methods for their approximate solution. In order to recapitulate the main results of those investigations we consider the following boundary value problem:

$$y''(t) - \frac{A_1(t)}{t}\, y'(t) - \frac{A_0(t)}{t^2}\, y(t) = f(t), \quad 0 < t \le 1, \tag{1a}$$

$$\mathcal{B}Y(0) = 0, \quad B_0Y(0) + B_1Y(1) = \beta, \;\; Y(t) = (y(t), y'(t))^T, \tag{1b}$$

where y and f are vector-valued functions of dimension n, A_0 and A_1 are $n \times n$ matrices, $\mathcal{B}, B_0$ and B_1 are constant matrices, such that $\mathcal{B}$ is a $q \times 2n$ matrix and B_0, B_1 are $p \times 2n$ matrices, with $q + p = 2n$, and finally, β is a p vector.

1.1 Analytical results for boundary value problems

Analytical properties of (1) have been studied in [2]. Here, we recapitulate main results concerning the existence of a unique bounded solution of (1a) and characterize the boundary conditions such a solution satisfies. For proofs and further details, cf. [2].

We apply the linear transformation

$$z(t) = (z_1(t), z_2(t))^T := (y(t), ty'(t))^T \tag{2}$$

to derive the first order system associated with (1a). This system reads

$$z'(t) = \frac{1}{t}\, M(t)z(t) + t\mathring{f}(t), \quad 0 < t \le 1, \tag{3}$$

where

$$M(t) := \begin{pmatrix} 0 & I_n \\ A_0(t) & I_n + A_1(t) \end{pmatrix}, \quad \mathring{f}(t) := \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

We denote by I_n the $n \times n$ identity matrix.

Let us assume that all entries of A_0 and A_1 are functions which are continuously differentiable[1] on $[0,1]$, $A_0, A_1 \in C^1 := C^1[0,1]$. Then using Taylor's theorem we can rewrite $A_0(t)$ and $A_1(t)$ and obtain

$$A_0(t) = A_0(0) + tC_0(t), \tag{4a}$$

$$A_1(t) = A_1(0) + tC_1(t), \tag{4b}$$

where C_0 and $C_1 \in C[0,1]$. Substitution of (4) into (3) implies

$$z'(t) = \frac{1}{t} M(0)z(t) + \mathring{C}(t)z(t) + t\mathring{f}(t), \quad 0 < t \le 1, \tag{5}$$

with

$$M := M(0) = \begin{pmatrix} 0 & I_n \\ A_0(0) & I_n + A_1(0) \end{pmatrix}, \quad \mathring{C}(t) := \begin{pmatrix} 0 & 0 \\ C_0(t) & C_1(t) \end{pmatrix}.$$

[1] More general, we denote by $C^p := C^p[0,1]$ the space of functions which are p times continuously differentiable on $[0,1]$.

As a first step in the analysis of (1a) we construct a general solution of (3) for the case of constant coefficient matrices $A_0(t) \equiv A_0$ and $A_1(t) \equiv A_1$, or equivalently, for the first order system

$$z'(t) = \frac{1}{t} M z(t) + t \overset{\circ}{f}(t), \quad 0 < t \le 1. \tag{6}$$

The general solution of (6) is given by

$$z(t) = t^M c + t^M \int_1^t s^{-M}\, s\, \overset{\circ}{f}(s)\, ds,$$

where

$$\Phi(t) = t^M := \exp(M \ln(t)),$$

and $\Phi(t)$ is the fundamental solution matrix of the homogeneous problem

$$V'(t) = \frac{1}{t} M V(t), \quad V(1) = I_{2n}.$$

Consequently, the solution z of (6) satisfies the terminal condition

$$z(1) = c, \quad c \in \mathbb{R}^{2n},$$

for any vector c. It is easily seen that in general, z is unbounded as $t \to 0$, more precisely, $z \in C^1(0,1]$. To see this we decompose the matrix M into its Jordan canonical form, $M := E^{-1} J E$, where E is the matrix of generalized eigenvectors of M. Then,

$$t^M = E^{-1} t^J E := E^{-1} \exp(J \ln(t)) E = E^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} (J \ln(t))^n E.$$

Let us assume that J consists of one Jordan block of dimension k,

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

then

$$t^J = t^\lambda \begin{pmatrix} 1 & \ln t & \frac{(\ln t)^2}{2!} & \dots & \frac{(\ln t)^{k-1}}{(k-1)!} \\ 0 & 1 & \ln t & \ddots & \frac{(\ln t)^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & \ddots & \frac{(\ln t)^{k-3}}{(k-3)!} \\ \vdots & \dots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

and the elements of t^J become unbounded[2] for $\lambda \le 0$ and $t \to 0$.

[2] For simplicity, we assume that all eigenvalues of M are real. All results can be easily generalized for the case when the eigenvalues of M are complex, see [2].

We now formulate the necessary conditions for z to be continuous on $[0,1]$.

Let X_0 be the eigenspace of M corresponding to the eigenvalue $\lambda = 0$ and X_+ be the invariant subspace associated with positive eigenvalues of M. Let σ_+ be the smallest positive eigenvalue, d_+ be the dimension of the largest Jordan block of M associated with σ_+, and d_0 be the dimension of the largest Jordan block of M associated with the eigenvalue $\lambda = 0$. Finally, let P be a projection onto $X_0 \bigoplus X_+$,

$$P := R + S,$$

where R and S are the projections onto X_0 and X_+, respectively, and

$$Q := I_{2n} - P.$$

We now have following results[3] for z:

Let $z \in C$ be a solution of (6) with $f \in C$. Then

$$Qz(0) = 0 \quad \text{and} \quad Sz(0) = 0.$$

For every $f \in C$ and a constant vector γ, there exists a unique solution $z \in C$, satisfying (6) and the terminal condition $Pz(1) = P\gamma$. This solution has the form

$$z(t) = (t^M P)P\gamma + t^M \int_1^t Ps^{-M}s\, \overset{\circ}{f}(s)\, ds + t^2 \int_0^1 Qs^{-M}s\, \overset{\circ}{f}(ts)\, ds. \tag{7}$$

The corresponding result for the solution y of (1a) with constant coefficient matrices A_0 and A_1 reads:

For every $f \in C$ and a constant vector γ, there exists a unique solution $y \in C$ satisfying the terminal condition

$$PY(1) = P\gamma.$$

This solution is given by the first n components of z,

$$y(t) = z_1(t).$$

Using the representation (7) and Banach's fixpoint theorem one can show the analogous result for the solution z of (5):

For every $f \in C$ and $C_0, C_1 \in C$, there exists a unique continuous solution of (5) subject to the terminal condition

$$Pz(1) = P\gamma.$$

This solution satisfies

$$Qz(0) = 0. \tag{8}$$

[3] For proofs, see [2]

Clearly, the solution of (1a) is given again by $y(t) = z_1(t)$. This solution satisfies the boundary value problem (1) if $q := \mathrm{rank} Q$ initial conditions given by $\mathcal{B}Y(0) = 0$ are equivalent to $Qz(0) = 0$ and imply that y is in $C[0,1]$. Moreover, $p := \mathrm{rank} P$ conditions given by $B_0Y(0) + B_1Y(1) = \beta$ have to provide p lineraly independent equations for the unknown vector $\tilde{P}\gamma$, where $\tilde{P}$ is the $p \times 2n$ matrix, consisting of the linearly independent rows of P.

Finally, let us show that for $y \in C^2$ the initial condition $Qz(0) = 0$ implies

$$\lim_{t\to 0} \Big(y''(t) - \frac{A_1}{t} y'(t) - \frac{A_0}{t^2} y(t) \Big) < \infty.$$

Using Taylor's theorem we rewrite (1a) and obtain

$$y''(t) - \frac{A_1}{t}\Big(y'(0) + ty''(\nu)\Big) - \frac{A_0}{t^2}\Big(y(0) + ty'(0) + \frac{t^2}{2} y''(\xi)\Big)$$

where $0 < \nu, \xi < t$. For $t \to 0$ this yields

$$\begin{aligned} &\lim_{t\to 0} \left(y''(t) - \frac{A_1}{t}\Big(y'(0) + ty''(\nu)\Big) - \frac{A_0}{t^2}\Big(y(0) + ty'(0) + \frac{t^2}{2} y''(\xi)\Big) \right) \\ &\quad = (I_n - A_1 - A_0/2) y''(0) < \infty, \end{aligned} \tag{9}$$

if

$$(A_1 + A_0) y'(0) = 0, \quad A_0 y(0) = 0. \tag{10}$$

Conditions (10) now follow if (8) holds, since in this case $z(0) = Rz(0)$ and $z'(0) = Uz'(0)$, where U is a projection on the eigenspace of M associated with the possible eigenvalue $\lambda = 1$. Consequently, this implies

$$\begin{aligned} Mz(0) &= MRz(0) = 0 \Leftrightarrow z_2(0) = 0, \;\; A_0 z_1(0) = A_0 y(0) = 0, \\ Mz'(0) &= MUz'(0) = z'(0) \Leftrightarrow z_1(0) = z_1(0), \;\; (A_0 + A_1) y'(0) = 0. \end{aligned}$$

This result holds also for the case that $\lambda = 0$ or/and $\lambda = 1$ are not eigenvalues of M and consequently, $R = 0$ or/and $U = 0$. In case of variable coefficient matrices $A_0(t) = A_0(0) + t^2 C_0(t)$ and $A_1(t) = A_1(0) + tC_1(t)$, where $C_0, C_1 \in C$ the same result holds with $A_0 := A_0(0)$ and $A_1 := A_1(0)$. This structure of $A_0(t)$ and $A_1(t)$ is motivated by the form of the Fréchet derivative of the nonlinear operator

$$D[y, y'](t) := y''(t) - \frac{A_1}{t} y'(t) - \frac{A_0}{t^2} y(t) - f(t, y(t), y'(t)).$$

1.2 Collocation schemes

In this section we give a brief overview of results on stability and convergence of a collocation method when applied to approximate the solution of (1). Detailed examinations and proofs can be found in [6].

Let us consider the equidistant partition of the interval $[0,1]$ with the stepsize h,

$$\Delta_h := \left\{ t_i = ih,\ i = 0(1)N,\quad h = \frac{1}{N} \right\},$$

where $i = 0(1)N$ is a compact notation for $i = 0,1,\dots,N-1,N$. On each subinterval $[t_i, t_{i+1}], i = 0(1)(N-1)$ we define the collocation points $t_{ik} := t_i + \mu_k h, k = 1(1)m$, by introducing constants $0 < \mu_1 < \ldots < \mu_m \le 1$. We denote by $\mathbb{P}_{m+1,\Delta}$ a class of functions whose elements are piecewise continuous on $[0,1]$ and which are polynomials of degree equal to $m+1$ in each subintervall $[t_i, t_{i+1}], i = 0(1)(N-1)$.

We appoximate y by a function $p \in \mathbb{P}_{m+1,\Delta} \cap C^1[0,1]$ satisfying the following collocation equations:

$$p''(t_{ik}) - \frac{A_1(t_{ik})}{t_{ik}}\, p'(t_{ik}) - \frac{A_0(t_{ik})}{t_{ik}^2}\, p(t_{ik}) = f(t_{ik}) \tag{11a}$$

with $i = 0(1)(N-1), k = 1(1)m$, and the boundary conditions

$$B_0 P(0) + B_1 P(1) = \beta,\quad P(t) = (p(t), p'(t))^T, \tag{11b}$$

$$\mathcal{B}P(0) = 0. \tag{11c}$$

We now formulate the stability and convergence properties of the collocation scheme.

Let the homogeneous boundary value problem (1) be uniquely solvable for any $f \in C, A_1 \in C^1, A_0 \in C^2$ and $\beta \in \mathbb{R}^p$. Then, if h is sufficiently small, there exists a unique solution $p \in \mathbb{P}_{m+1,\Delta} \cap C^1$ of (11a,11b) with $\mathcal{B}P(0) = \delta, \delta \in \mathbb{R}^q$ and the following estimate holds:

$$|p^{(i)}(t)| \le \text{const}\ \{|\delta|\,|\ln h|^{d_0-1} + |\beta| + |f(\Delta)|\},\quad i = 0,1,$$

where $f(\Delta) := (f(t_0), f(t_1), \dots, f(t_N))^T$, and

$$|y(t)| = |(y_1(t), \dots, y_N(t))^T| := \max\{|y_1(t)|, \dots, |y_N(t)|\},$$

for a vector depending on t.

Let $f \in C^m, A_1 \in C^{m+1}, A_0 \in C^{m+2}$. Let $y \in C^1[0,1] \cap C^{m+2}(0,1]$ and $p \in \mathbb{P}_{m+1,\Delta} \cap C^1$ be the unique solution of (1) and (11), respectively. Then, for h sufficiently small, the following estimates hold:

$$|y^{(i)}(t) - p^{(i)}(t)| \le \begin{cases} \text{const}\ h^{\sigma_+}|\ln h|^{d_+-1}, & 1 < \sigma_+ < m, \\ \text{const}\ h^m|\ln h|^{d_+-1}, & m \le \sigma_+ < m+2, \\ \text{const}\ h^m|\ln h|^{d_+}, & \sigma_+ = m+2, \\ \text{const}\ h^m, & \sigma_+ > m+2, \end{cases}$$

for $0 \le t \le 1,\ i = 0,1$. Furthermore, if $y \in C^{m+2}$, then[4]

$$|y^{(i)}(t) - p^{(i)}(t)| \le \text{const}\ h^m,\ \ 0 \le t \le 1,\ i = 0,1.$$

[4]Clearly, for $y \in C^{m+2}$ it is sufficient that $M(0)$ has no positive eigenvalues, or the only positive eigenvalues are natural numbers and the associated Jordan blocks are diagonal.

It is well-known that a suitable choice of the collocation points $\mu_1, \ldots, \mu_m$ leads to the so-called superconvergence of the scheme at the gridpoints t_i, $i = 0(1)N$. If

$$\int_0^1 s^k w(s)\, ds = 0, \quad k = 0(1)\nu,\ \nu < m,$$

where $w(s) := (s - \mu_1)\ldots(s - \mu_m)$ holds and the solution y is smooth enough, then the following error estimate holds:

$$|y^{(l)}(t_i) - p^{(l)}(t_i)| \leq \text{const } h^{m+\nu+1},\ \ i = 0(1)N,\ \ l = 0, 1.$$

For the singular case we have:
Let $f, C_1 \in C^{m+1}, C_0 \in C^{m+2}$ and $\sigma_+ > m + 3$. Let $y \in C^{m+3}$ and $p \in \mathbb{P}_{m+1,\Delta} \cap C^1$ be solutions of (1) and (11), respectively. Then the estimate

$$|y^{(i)}(t) - p^{(i)}(t)| \leq \text{const } h^{m+1},\ \ 0 \leq t \leq 1,\ \ i = 0, 1,$$

holds. In general, this result cannot be improved.

To see this let us consider a special model problem[5] of the form

$$y''(t) - \frac{A_1(t)}{t}\, y'(t) - \frac{A_0(t)}{t^2}\, y(t) = f(t), \quad 0 < t \leq 1, \tag{12a}$$

$$y(0) = 0, \quad B_1 Y(1) = \beta, \tag{12b}$$

$$A_1(t) = A_1 + tC_1(t), \quad A_0(t) = A_0 + t^2 C_0(t), \tag{12c}$$

where A_1 and A_0 are diagonal, $A_0 - A_1 = 2I_n$ and the elements of A_0 are integers $a_{0,i} \geq 1, i = 1(1)n$. This means that the eigenvalues of $M(0)$ are $\lambda_{1i} = -1, i = 1(1)n,\ \lambda_{2i} = a_{0,i}, i = 1(1)n$.

Let f, C_1 and $C_0 \in C^{m+\nu+1}, \nu \geq 1$, let B_1 be an $n \times 2n$ matrix with $\text{rank} B_1 = n$ and $\beta \in \mathbb{R}^n$. In this case $q = p = n$. Furthermore, let the Jordan matrix J be diagonal and set $\sigma := \min_i a_{0,i}$. Then the following result holds:
If $y \in C^{m+\nu+3}$ is a unique solution of (12) and $p \in \mathbb{P}_{m+1,\Delta} \cap C^1$ its approximation defined by the associated collocation scheme (11), then

$$|y(t_i) - p(t_i)| \leq \text{const } h^{m+\nu+1}((t_i + h)^{1-\nu} + t_i^{\sigma}),\ \ i = 0(1)n,$$
$$|y'(t_i) - p'(t_i)| \leq \text{const } h^{m+\nu+1}((t_i + h)^{-\nu} + t_i^{\sigma-1}),\ \ i = 0(1)n.$$

This result means that near to the singularity the superconvergence order cannot be guaranteed, in general.

1.3 Finite difference methods

In this section we discuss stability and convergence of a certain finite difference scheme when used to approximate the solution of (1). For details see [3].

[5] The form and the data of this model are typical for applications from mechanics.

On the equidistant grid with the stepsize h, defined in the last section, the standard three-point discretization for (1) reads

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}-\frac{A_1(t_i)}{t_i}\frac{y_{i+1}-y_{i-1}}{2h}-\frac{A_0(t_i)}{t_i^2}y_i=f(t_i), \tag{13a}$$

for $i=1(1)(N-1)$,

$$\mathcal{B}Y_0=0, \quad B_0Y_0+B_1Y_N=\beta, \tag{13b}$$

where $Y_0:=(y_0,y_0')^T, Y_N:=(y_N,y_N')^T$. By y_0',y_N' we denote the approximations for $y'(0)$ and $y'(1)$, respectively.

To approximate $y'(1)$ we choose $y_N':=(y_{N+1}-y_{N-1})/2h$ and complete the difference scheme (13a) by one more equation for $i=N$. The choice of y_0' is not that simple, because we cannot include the point $t_{-1}=-h$ into the scheme. However, we can remedy the situation if the solution y is smooth enough.

Let $y\in C^3$. Then

$$(y(h)-y(0))/h=y'(0)+\frac{h}{2}y''(0)+O(h^2).$$

We now obtain from (9)

$$A(0)y''(0):=(I_n-A_1(0)-A_0(0)/2)y''(0)=f(0)$$

and consequently, if $A(0)^{-1}$ exists, the natural approximation for $y'(0)$ is

$$y_0':=(y_1-y_0)/h-hA(0)^{-1}f(0)/2.$$

If the last expression is not well-defined, another approximation must be taken. For instance,

$$y_0':=(-y_2+4y_1-3y_0)/2h.$$

Both approximations y_0' for $y'(0)$ are $O(h^2)$ formulas if $y\in C^3$.

We now formulate the stability and convergence results.
If the homogenous boundary problem (1) has only the trivial solution, then, for each $t_{i_\star}=:\varepsilon>0$, there exists a stepsize $h(\varepsilon)>0$ such that, for every $h\le h(\varepsilon)$, the system (13a) subject to the boundary conditions

$$\mathcal{B}Y_{i_\star}=\delta, \quad B_0Y_{i_\star}+B_1Y_N=\beta, \tag{14}$$

has a unique solution for each f and the following estimate holds:

$$|y_{\Delta_{i_\star}}|\le \text{const }\{|\delta|\,|\ln h|^{d_0-1}+|\beta|+|f(\Delta_{i_\star})|\},$$

where $y_{\Delta_{i_\star}}:=(y_{i_\star},y_{i_\star+1},\dots,y_N)^T$ and $f(\Delta_{i_\star}):=(f(t_{i_\star}),f(t_{i_\star+1}),\dots,f(t_N))^T$.

Let $y_{\Delta_{i_\star}}$ be a solution of (13a) subject to the boundary conditions (14). For $f, C_1 \in C^2, C_0 \in C^3$, and for a sufficiently small stepsize h the following estimate for the global error holds:

$$|y_{\Delta_{i_\star}} - y(\Delta_{i_\star})| \le \begin{cases} \text{const } h^{\sigma_+} |\ln h|^{d_+-1}, & 0 < \sigma_+ < 2, \\ \text{const } h^2 \{ |\ln h|^{d_+} + |\ln h|^{d_0-1} \}, & \sigma_+ = 2, \\ \text{const } h^2 |\ln h|^{d_0-1}, & \sigma_+ > 2 \text{ or } S = 0. \end{cases}$$

Here, an additional remark is in place. The stability and convergence results differ from the classical ones, since they hold on a part of the grid $\Delta_{i_\star}$, namely for points t_i, $i \ge i_\star$, only. It is important to mention that the restriction $i \ge i_\star$, where $i_\star$ depends merely on the problem data, is sufficient but *not necessary* for a stability result to follow. Moreover, for h chosen appriopriately small we can choose $t_{i_\star}$ arbitrarily close to zero. In practice this restriction has not caused difficulties yet, and the method performed well[6] for $i_\star = 0$.

2 Analytical Results for Initial Value Problems

The experimental results reported in Section 3 on the convergence behavior of multistep methods have been obtained for scalar *initial value problems* with constant coefficients.

The basic analytical properties of singular problems for initial value problems have not been studied yet, so some remarks on existence, smoothness and uniqueness of solutions for such problems have to be made. They can be derived using techniques developed for boundary value problems in Section 1.

We restrict our attention to typical model cases, serving as test problems in Section 3. We consider the following scalar differential equation of second order, subject to initial conditions or terminal conditions, respectively:

$$y''(t) - \frac{a_1}{t} y'(t) - \frac{a_0}{t^2} y(t) = f(t), \quad 0 < t \le 1, \tag{15a}$$

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{15b}$$

$$y(1) = \alpha, \quad y'(1) = \beta, \tag{15c}$$

where a_0 and a_1 are real numbers and $f \in C^m[0,1]$, $m \ge 0$. We first construct the general solution of (15a) and derive conditions for (15a,15b) and (15a,15c) to have a unique smooth solution. It turns out that the values α and β have to be chosen in a proper way to ensure the solution y to be continuous on the whole interval $[0,1]$. We use the linear transformation (2), $z(t) = (z_1(t), z_2(t))^T := (y(t), ty'(t))^T$, in order to obtain the associated first order differential equation for z,

$$z'(t) = \frac{1}{t} M z(t) + t\mathring{f}(t), \quad 0 < t \le 1, \tag{16}$$

where

$$M := \begin{pmatrix} 0 & 1 \\ a_0 & 1 + a_1 \end{pmatrix}, \quad \mathring{f}(t) := \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{17}$$

[6] Also for linear and nonlinear systems.

Let J be the Jordan canonical form of the matrix M and E the matrix of generalized eigenvectors of M. Then M can be represented in the form $M = EJE^{-1}$ and the substitution $v(t) := E^{-1}z(t)$, $\mathring{g}(t) := E^{-1}\mathring{f}(t)$ leads to the following system for v :

$$v'(t) = \frac{1}{t}\, Jv(t) + t\,\mathring{g}(t). \tag{18}$$

2.1 General solution of the homogeneous problem

The fundamental solution matrix of the homogeneous problem

$$V'(t) = \frac{1}{t}\, JV(t), \quad V(1) = I_2$$

is given by $V(t) = t^J$. Let us assume that M has two real eigenvalues λ_1 and λ_2. For $\lambda_1 \neq \lambda_2$ we obtain

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \Rightarrow \quad t^J = \begin{pmatrix} t^{\lambda_1} & 0 \\ 0 & t^{\lambda_2} \end{pmatrix},$$

and for[7] $\lambda_1 = \lambda_2 = \lambda$,

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \Rightarrow \quad t^J = t^\lambda \begin{pmatrix} 1 & \ln t \\ 0 & 1 \end{pmatrix}$$

holds. Finally, for an arbitrary vector $c \in \mathbb{R}^2$ the general solution v_h of the homogeneous problem (18) has the form $v_h(t) = t^J c$. This means that the general solution y_h of the homogeneous equation (15a) reads

$$y_h(t) = z_1(t) = (Ev_h(t))_1 = \begin{cases} c_1 t^{\lambda_1} + c_2 t^{\lambda_2}, & \text{if } \lambda_1 \neq \lambda_2, \\ c_1 t^\lambda + c_2 t^\lambda \ln t, & \text{if } \lambda_1 = \lambda_2 = \lambda, \end{cases}$$

with two real constants c_1 and c_2.

2.2 Particular solution of the inhomogeneous problem

We now use the fundamental solution matrix to represent a particular solution v_p of the inhomogeneous problem (18) and take again the first component of Ev_p to obtain the solution y_p of (15a):

$$v_p(t) = t^J \int_1^t s^{-J} s\,\mathring{g}(s)\, ds, \quad v_p(1) = 0. \tag{19}$$

We show that for $f \in C[0,1]$ we can always construct $v_p \in C[0,1]$ and consequently $y_p \in C[0,1]$.

[7] In the scalar case, if M has a double eigenvalue λ, then there exists only one eigenvector associated with λ and therefore the matrix J cannot be diagonal.

Case 1. The eigenvalues of M are nonpositive
In this case we rewrite (19) and obtain

$$\begin{aligned} v_p(t) &= t^J \int_1^t s^{-J} s\, E^{-1} \mathring{f}(s)\, ds \\ &= t^J \left(\int_1^0 s^{-J} s\, E^{-1} \mathring{f}(s)\, ds + \int_0^t s^{-J} s\, E^{-1} \mathring{f}(s)\, ds \right) \\ &= t^J \gamma + t^J \int_0^t s^{-J} s\, E^{-1} \mathring{f}(s)\, ds. \end{aligned}$$

We treat the term $t^J\gamma$ as a contribution to the general solution $v_h(t)$ of the homogeneous problem and choose the integral expression for $v_p(t)$,

$$v_p(t) = t^2 \int_0^1 s^{-J} s\, E^{-1} \mathring{f}(ts)\, ds. \tag{20}$$

We substitute (20) into (18) to deduce

$$v_p'(t) = Jt \int_0^1 s^{-J} s\, E^{-1} \mathring{f}(ts)\, ds + tE^{-1}\mathring{f}(t). \tag{21}$$

We denote by $|A(t)|$ the infinity norm for a matrix $A(t) \in \mathbb{R}^{2\times 2}$ depending on t,

$$|A(t)| := \|A(t)\|_\infty,$$

and by $\|y\|_\infty$ the infinity norm for a vector-valued function y,

$$\|y\|_\infty := \max_{0\le t\le 1} |y(t)|.$$

For $v_p(t)$ we now have

$$|v_p(t)| \le \text{const}\ t^2\, \|f\|_\infty$$

and

$$|v_p'(t)| \le \text{const}\ t\, \|f\|_\infty.$$

Moreover,

$$v_p(t) \in C^2[0,1], \;\; v_p(0) = v_p'(0) = 0.$$

This means that if the right-hand side f is continuous and the matrix M has nonpositive real eigenvalues it is always possible to construct a particular solution $y_p(t) = (Ev_p(t))_1$ of (15a) which is in $C^2[0,1]$ and satisfies $y_p(0) = y_p'(0) = 0$. Clearly, the general solution of (15a) is

$$y(t) = y_p(t) + y_h(t) = y_p(t) + \begin{cases} c_1 t^{\lambda_1} + c_2 t^{\lambda_2}, & \text{if } \lambda_1 \ne \lambda_2, \\ c_1 t^{\lambda} + c_2 t^{\lambda} \ln t, & \text{if } \lambda_1 = \lambda_2 = \lambda. \end{cases} \tag{22}$$

Boundary conditions: It is clear from (22) that in general $y \in C^2(0,1]$ only. The following list provides sufficient conditions for $y \in C^2[0,1]$ and consequently, for y to be bounded on the whole interval $[0,1]$:

$$\lambda_1 < \lambda_2 < 0: \qquad y(0) = y'(0) = 0 \Rightarrow y(t) = y_p(t),$$

$$\lambda_1 = \lambda_2 < 0: \qquad y(0) = y'(0) = 0 \Rightarrow y(t) = y_p(t),$$

$$\lambda_1 < \lambda_2 = 0: \qquad y(0) = \alpha,\ y'(0) = 0 \Rightarrow y(t) = \alpha + y_p(t),$$

$$\lambda_1 = \lambda_2 = 0: \qquad y(0) = \alpha,\ y'(0) = 0 \Rightarrow y(t) = \alpha + y_p(t).$$

Case 2. The eigenvalues of M are positive

We use (19) to derive the estimate for $v_p(t)$. Clearly,

$$|v_p(t)| \le \int_1^t \left| \left(\frac{t}{s}\right)^J \right| s\,ds \|E^{-1}\|_\infty \|f\|_\infty. \tag{23}$$

Case 2.1. $0 < \lambda_1 < \lambda_2$: It follows from (23) that

$$|v_p(t)| \le \begin{cases} \text{const } t^{\lambda_1}|t^{-\lambda_1+2} - 1| \|f\|_\infty = \text{const } |t^2 - t^{\lambda_1}| \|f\|_\infty, & \lambda_1 \ne 2, \\ \text{const } t^2 |\ln t - 1| \|f\|_\infty = \text{const } |t^2 \ln t - t^2| \|f\|_\infty, & \lambda_1 = 2, \end{cases}$$

and consequently,

$$|v_p(t)| \le \begin{cases} \text{const } t^{\lambda_1} \|f\|_\infty, & \lambda_1 < 2, \\ \text{const } t^2(1 + |\ln t|) \|f\|_\infty, & \lambda_1 = 2, \\ \text{const } t^2 \|f\|_\infty, & \lambda_1 > 2. \end{cases}$$

Case 2.2. $\lambda = \lambda_1 = \lambda_2$: Again, from (23) we obtain

$$|v_p(t)| \le \begin{cases} \text{const } t^{\lambda}(1 + |\ln t|) \|f\|_\infty, & \lambda < 2, \\ \text{const } t^2(1 + |\ln t|^2) \|f\|_\infty, & \lambda = 2, \\ \text{const } t^2 \|f\|_\infty, & \lambda > 2. \end{cases}$$

Also, in both cases $v_p \in C[0,1]$ and $v_p(0) = 0$ holds.

This means that if $f \in C[0,1]$ and the eigenvalues of M are positive the solution $v_p(t)$, given by (19) is continuous on $[0,1]$ and therefore the particular solution $y_p(t) = (Ev_p(t))_1$ of (15a) is continuous on $[0,1]$. Clearly the general solution of (15a) is

$$y(t) = y_p(t) + y_h(t) = y_p(t) + \begin{cases} c_1 t^{\lambda_1} + c_2 t^{\lambda_2}, & \lambda_1 \ne \lambda_2, \\ c_1 t^{\lambda} + c_2 t^{\lambda} \ln t, & \lambda = \lambda_1 = \lambda_2. \end{cases} \tag{24}$$

Boundary conditions: From

$$z(t) = \begin{pmatrix} y(t) \\ ty'(t) \end{pmatrix} = t^M z(1) + Ev_p(t)$$

we immediately see that the unique bounded solution of (15a,15c) is

$$y(t) = z_1(t) = \left(t^M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + Ev_p(t) \right)_1 . \tag{25}$$

To see that y satisfies the terminal conditions (15c) we evaluate (25) for $t = 1$ and obtain

$$y(1) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_1 = \alpha.$$

According to (16),

$$y'(t) = z_1'(t) = \left(\frac{1}{t} M \left(t^M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + Ev_p(t) \right) + t\mathring{f}(t) \right)_1$$

holds, the evaluation for $t = 1$ implies

$$y'(1) = \left(\begin{pmatrix} 0 & 1 \\ a_0 & 1 + a_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)_1 = \beta.$$

Finally, we formulate smoothness properties for y. It can be easily seen from the form of y that the following result holds:
In case of simple eigenvalues, $y \in C^1[0,1]$ for $\lambda_1 \geq 1$ and $y \in C^2[0,1]$ for $\lambda_1 \geq 2$. If λ is a twofold eigenvalue of M, then $y \in C^1[0,1]$ for $\lambda > 1$ and $y \in C^2[0,1]$ for $\lambda > 2$.

The results presented in Section 2 can now be summarized as follows:

Let $\lambda_1 \leq \lambda_2 < 0$. Then, for every $f \in C^m[0,1]$, there exists a unique solution $y \in C[0,1]$ of (15a,15b) if $\alpha = \beta = 0$. This solution satisfies $y \in C^{m+2}[0,1]$.

Let $\lambda_1 \leq \lambda_2 = 0$. Then, for every $f \in C^m[0,1]$ and $\alpha \in \mathbb{R}$, there exists a unique solution $y \in C[0,1]$ of (15a,15b) if $\beta = 0$. This solution satisfies $y \in C^{m+2}[0,1]$.

Let $0 < \lambda_1 \leq \lambda_2$. Then, for every $f \in C^m[0,1]$ and $\alpha, \beta \in \mathbb{R}$, there exists a unique solution $y \in C[0,1]$ of (15a,15c). This solution satisfies $y \in C^{m+2}[0,1]$ if $\lambda_1 > m+2$, $y \in C^{m+1}[0,1]$ if $\lambda_1 > m+1$ and $y \in C^m[0,1]$ if $\lambda_1 > m$.

3 Multistep Methods

We present now the experimental results showing how standard multistep methods perform when applied to solve numerically the following scalar problem:

$$y''(t) - \frac{a_1}{t} y'(t) - \frac{a_0}{t^2} y(t) = f(t), \quad 0 < t \leq 1, \tag{26a}$$

subject to either initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta, \tag{26b}$$

or final conditions

$$y(1) = \alpha, \qquad y'(1) = \beta, \tag{26c}$$

whose solution y is assumed to be unique and smooth.

De Hoog and Weiss have shown in [1] that in case of singular problems of the first order the root condition is no longer sufficient for the stability of the linear multistep methods. We used an experimental approach to find out about the stability and convergence properties of multistep methods and to provide the basis for the theoretical investigation to follow. Especially, we were interested to find out how the eigenvalues of the matrix M and the smoothness of y influence the performance of the methods.

Let us again consider the equidistant grid Δ_h with the stepsize h, and the intitial value problem

$$\begin{aligned} &y''(t) = g(t, y(t), y'(t)), \quad 0 < t \le 1, \\ &y(0) = \alpha, \qquad y'(0) = \beta. \end{aligned}$$

In order to solve this problem numerically we use, see [8], the explicit Adams-Bashforth formula. For $j = 0(1)(n-2)$ we have

$$\begin{aligned} y_{j+2} &= y_{j+1} + hy'_{j+1} + \frac{h^2}{6}(-g_j + 4g_{j+1}), \\ y'_{j+2} &= y'_{j+1} + \frac{h}{2}(-g_j + 3g_{j+1}), \\ y_0 &= \alpha, \quad y_1 = \alpha_1, \\ y'_0 &= \beta, \quad y'_1 = \beta_1, \end{aligned}$$

where the starting values α_1 and β_1 are obtained from an appropriate one-step method and $g_j = g(t_j, y_j, y'_j)$.

We also use the implicit Adams-Moulton formula; $j = 0(1)(n-1)$:

$$\begin{aligned} y_{j+1} &= y_j + hy'_j + \frac{h^2}{6}(2g_j + g_{j+1}), \\ y'_{j+1} &= y'_j + \frac{h}{2}(g_j + g_{j+1}), \\ y_0 &= \alpha, \\ y'_0 &= \beta. \end{aligned}$$

For smooth initial value problems without singularity these procedures are stable second order methods.[8] Additionally, the corresponding third order methods have been considered.

[8] Provided that the starting values $y_1 = \alpha_1$ and $y'_1 = \beta_1$ are $O(h^2)$ approximations for $y(t_1)$ and $y'(t_1)$, respectively.

4 Experimental Results

The solutions of around 2000 test problems have been computed. In most cases the methods performed well showing the classical behavior. In certain cases though, the classical convergence order has not been observed. Here we show results typically observed while solving singular problems and formulate open questions to be answered in the course of theoretical study.

We investigate models of the form

$$y''(t) = \frac{\lambda_1 + \lambda_2 - 1}{t} y'(t) - \frac{\lambda_1 \lambda_2}{t^2} y(t) + f(t), \quad 0 < t \le 1,$$

where λ_1 and λ_2 are the eigenvalues of the matrix M. All computations were carried out on an Intel 80486 DX2 machine in double precision.[9]

In tables we present the estimates for the order p of the method and the error constant c obtained in the following way:

Let us denote by $y_n(h)$ the approximation for $y(1)$ obtained on an equidistant grid with the stepsize h and assume that the global error δ at $t = 1$ can be written in the form

$$y(1) - y_n(h) \approx c\,h^p. \tag{27}$$

We calculate two different approximations $y_n(h_1)$ and $y_n(h_2)$ with different stepsizes h_1 and h_2 and have

$$y(1) - y_n(h_1) \approx c\,h_1^p$$

and

$$y(1) - y_n(h_2) \approx c\,h_2^p.$$

Consequently,

$$p \approx \frac{\ln\left(\frac{|y(1)-y_n(h_1)|}{|y(1)-y_n(h_2)|}\right)}{\ln\left(\frac{|h_1|}{|h_2|}\right)}$$

and

$$c \approx \frac{y(1) - y_n(h_1)}{h_1^p}.$$

The stepsizes $h_1 > h_2$ are chosen to satisfy $\frac{h_1}{h_2} \in \mathbb{N}$.

When integrating from $t = 1$ towards $t = 0$ and using an implicit method, we choose a point $0 < h^* < 1$, close to the singularity, to monitor the global error and to estimate the order of convergence. We calculate two different approximations $y_{n-j_1}(h_1)$ and $y_{n-j_2}(h_2)$ with different stepsizes h_1 and h_2, which satisfy $h_1 j_1 = h_2 j_2 = h^*$. Thus we have

$$y(h^*) - y_{n-j}(h) \approx c\,h^p$$

instead of (27), where $hj = h^*$. If explicit methods are used then $h^* = 0$.

[9] Relative machine accuracy 10^{-16}.

4.1 Example 1: $\lambda_1 < \lambda_2 = 0$

The initial value problem under consideration reads

$$y''(t) = \frac{\lambda_1 - 1}{t} y'(t) + (49 - 7\lambda_1)t^5, \quad 0 < t \leq 1,$$
$$y(0) = 0, \quad y'(0) = 0.$$

Independently of the choice of λ_1 the exact solution[10] is $y(t) = t^7$. Moreover, $\lambda_1 = -1, -5$ and -10.

For this singular problem all formulas retain their classical convergence orders. Order two or three, respectively, is observed independently of the magnitude of the eigenvalue λ_1 and the error constants remain moderately sized. In Table 1 convergence orders and error constants for the Adams-Moulton method are listed.

h	$p_{(-1,0)}$	$c_{(-1,0)}$	$p_{(-5,0)}$	$c_{(-5,0)}$	$p_{(-10,0)}$	$c_{(-10,0)}$
$1/5$	2.0542	$-2.6 \cdot 10^{+00}$	2.0985	$-1.8 \cdot 10^{+00}$	2.1558	$-1.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-1}$	2.0142	$-2.4 \cdot 10^{+00}$	2.0264	$-1.5 \cdot 10^{+00}$	2.0429	$-1.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-2}$	2.0035	$-2.3 \cdot 10^{+00}$	2.0067	$-1.4 \cdot 10^{+00}$	2.0110	$-9.7 \cdot 10^{+00}$
$1/5 \cdot 2^{-3}$	2.0009	$-2.3 \cdot 10^{+00}$	2.0016	$-1.4 \cdot 10^{+00}$	2.0027	$-9.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	2.0002	$-2.3 \cdot 10^{+00}$	2.0004	$-1.4 \cdot 10^{+00}$	2.0006	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	2.0000	$-2.3 \cdot 10^{+00}$	2.0001	$-1.4 \cdot 10^{+00}$	2.0001	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	2.0000	$-2.3 \cdot 10^{+00}$	2.0000	$-1.4 \cdot 10^{+00}$	2.0000	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	2.0000	$-2.3 \cdot 10^{+00}$	2.0000	$-1.4 \cdot 10^{+00}$	2.0000	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-8}$	2.0000	$-2.3 \cdot 10^{+00}$	2.0000	$-1.4 \cdot 10^{+00}$	2.0000	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-9}$	1.9999	$-2.3 \cdot 10^{+00}$	1.9999	$-1.3 \cdot 10^{+00}$	1.9999	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-10}$	1.9999	$-2.3 \cdot 10^{+00}$	1.9999	$-1.3 \cdot 10^{+00}$	1.9998	$-9.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-11}$	2.0003	$-2.3 \cdot 10^{+00}$	2.0006	$-1.4 \cdot 10^{+00}$	2.0010	$-9.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-12}$	2.0026	$-2.3 \cdot 10^{+00}$	2.0048	$-1.4 \cdot 10^{+00}$	2.0076	$-1.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-13}$	1.9795	$-1.8 \cdot 10^{+00}$	1.9600	$-9.1 \cdot 10^{+00}$	1.9350	$-4.6 \cdot 10^{+00}$
$1/5 \cdot 2^{-14}$	1.8373	$-3.7 \cdot 10^{+00}$	1.7254	$-6.4 \cdot 10^{-01}$	1.5970	$-1.0 \cdot 10^{-02}$

Table 1: Convergence of the Adams-Moulton method; $\lambda_1 = -1, -5, -10$, $\lambda_2 = 0$.

This example is representative for all models with eigenvalues $\lambda_1, \lambda_2 \leq 0$. Figure 1 shows the convergence orders for $-10 \leq \lambda_1, \lambda_2 \leq 0$. For $\lambda_1 \approx \lambda_2$ the convergence order slightly drops.

[10] In this case the general solution if the homogeneous problem is trivial.

Figure 1: Convergence orders of the Adams-Bashforth method.

4.2 Example 2.1: $\lambda_2 > \lambda_1 > 0$

Here we discuss the problem

$$y''(t) = \frac{\lambda_1 + \lambda_2 - 1}{t} y'(t) - \frac{\lambda_1 \lambda_2}{t^2} y(t), \quad 0 < t \leq 1,$$
$$y(1) = 2, \quad y'(1) = \lambda_1 + \lambda_2.$$

Its exact solution is $y(t) = t^{\lambda_1} + t^{\lambda_2}$, where $\lambda_1 = 1$ and $\lambda_2 = 3, 5$.

Tables 2 to 5 show the absolute errors and convergence orders for the Adams-Bashforth formula for this case. For $\lambda_2 = 3$ the convergence is faster than in classical case, which may be explained by a simple solution structure. For $\lambda_2 = 5$ classical results, similar to those in Example 1, can be observed. The same holds for $\lambda_1 = \lambda_2 > 0$ and $\lambda_1 > \lambda_2 = 0$.

A surprising result is a very fast convergence in the case $\lambda_2 = 5$ and $h^* = 0$.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5 \cdot 2^{-1}$	$1.7 \cdot 10^{-04}$	3.096	$-2.1 \cdot 10^{-01}$	$3.6 \cdot 10^{-02}$	1.551	$1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$2.0 \cdot 10^{-05}$	3.051	$-1.8 \cdot 10^{-01}$	$1.2 \cdot 10^{-03}$	1.844	$3.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-3}$	$2.4 \cdot 10^{-06}$	3.026	$-1.7 \cdot 10^{-01}$	$3.5 \cdot 10^{-03}$	1.932	$4.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$2.9 \cdot 10^{-07}$	3.013	$-1.6 \cdot 10^{-01}$	$9.1 \cdot 10^{-04}$	1.968	$5.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$3.6 \cdot 10^{-07}$	3.006	$-1.5 \cdot 10^{-01}$	$2.3 \cdot 10^{-05}$	1.984	$5.5 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$4.5 \cdot 10^{-08}$	3.003	$-1.5 \cdot 10^{-01}$	$5.9 \cdot 10^{-05}$	1.992	$5.8 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$5.7 \cdot 10^{-09}$	3.001	$-1.5 \cdot 10^{-01}$	$1.4 \cdot 10^{-06}$	1.996	$5.9 \cdot 10^{+00}$
$1/5 \cdot 2^{-8}$	$7.1 \cdot 10^{-10}$	2.996	$-1.4 \cdot 10^{-01}$	$3.7 \cdot 10^{-06}$	1.998	$6.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-9}$	$8.9 \cdot 10^{-11}$	3.100	$-3.2 \cdot 10^{+00}$	$9.3 \cdot 10^{-07}$	1.999	$6.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-10}$	$1.0 \cdot 10^{-12}$	4.970	$-2.8 \cdot 10^{+06}$	$2.3 \cdot 10^{-08}$	1.999	$6.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-11}$	$3.3 \cdot 10^{-13}$	-3.150	$-7.7 \cdot 10^{-26}$	$5.8 \cdot 10^{-08}$	2.000	$6.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-12}$	$2.9 \cdot 10^{-13}$	-0.598	$-7.7 \cdot 10^{-15}$	$1.4 \cdot 10^{-09}$	2.001	$6.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-13}$	$4.4 \cdot 10^{-12}$	-1.297	$-4.6 \cdot 10^{-18}$	$3.6 \cdot 10^{-09}$	1.978	$4.8 \cdot 10^{+00}$
$1/5 \cdot 2^{-14}$	$1.0 \cdot 10^{-12}$	-0.671	$5.5 \cdot 10^{-15}$	$9.2 \cdot 10^{-10}$	1.900	$2.0 \cdot 10^{-01}$
$1/5 \cdot 2^{-15}$	$1.7 \cdot 10^{-12}$	-1.334	$1.9 \cdot 10^{-19}$	$2.4 \cdot 10^{-11}$	4.877	$6.6 \cdot 10^{+15}$

Table 2: Convergence of the Adams-Bashforth method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0.1$.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5 \cdot 2^{-3}$	$6.0 \cdot 10^{-06}$	3.026	$-4.2 \cdot 10^{-01}$	$8.9 \cdot 10^{-04}$	1.931	$1.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-4}$	$7.4 \cdot 10^{-07}$	3.013	$-4.0 \cdot 10^{-01}$	$2.3 \cdot 10^{-05}$	1.968	$1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$9.2 \cdot 10^{-08}$	3.006	$-3.9 \cdot 10^{-01}$	$5.9 \cdot 10^{-05}$	1.984	$1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$1.1 \cdot 10^{-09}$	3.003	$-3.8 \cdot 10^{-01}$	$1.5 \cdot 10^{-06}$	1.992	$1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$1.4 \cdot 10^{-10}$	2.999	$-3.7 \cdot 10^{-01}$	$3.7 \cdot 10^{-06}$	1.996	$1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$1.7 \cdot 10^{-11}$	2.979	$-3.2 \cdot 10^{-01}$	$9.5 \cdot 10^{-07}$	1.998	$1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$2.2 \cdot 10^{-12}$	3.565	$-3.2 \cdot 10^{+01}$	$2.3 \cdot 10^{-08}$	1.999	$1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-10}$	$1.9 \cdot 10^{-13}$	0.788	$-1.6 \cdot 10^{-10}$	$5.9 \cdot 10^{-08}$	1.999	$1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-11}$	$1.1 \cdot 10^{-13}$	-1.664	$2.3 \cdot 10^{-20}$	$1.4 \cdot 10^{-09}$	2.001	$1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-12}$	$3.5 \cdot 10^{-12}$	-0.745	$-2.1 \cdot 10^{-16}$	$3.7 \cdot 10^{-09}$	2.007	$1.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-13}$	$5.9 \cdot 10^{-12}$	-1.231	$-1.2 \cdot 10^{-18}$	$9.2 \cdot 10^{-10}$	1.904	$5.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-14}$	$1.3 \cdot 10^{-12}$	-0.762	$2.4 \cdot 10^{-16}$	$2.4 \cdot 10^{-11}$	1.584	$1.5 \cdot 10^{-03}$
$1/5 \cdot 2^{-15}$	$2.3 \cdot 10^{-12}$	-1.239	$8.0 \cdot 10^{-18}$	$8.2 \cdot 10^{-11}$	0.955	$7.8 \cdot 10^{-06}$

Table 3: Convergence of the Adams-Bashforth method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0.025$.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5 \cdot 2^{-5}$	$2.3 \cdot 10^{-09}$	3.006	$-9.7 \cdot 10^{-02}$	$1.4 \cdot 10^{-06}$	1.984	$3.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$2.8 \cdot 10^{-10}$	3.003	$-9.6 \cdot 10^{-02}$	$3.7 \cdot 10^{-06}$	1.992	$3.7 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$3.5 \cdot 10^{-10}$	2.993	$-9.0 \cdot 10^{-02}$	$9.4 \cdot 10^{-07}$	1.996	$3.7 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$4.4 \cdot 10^{-11}$	2.910	$-4.9 \cdot 10^{-02}$	$2.3 \cdot 10^{-08}$	1.998	$3.8 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$5.9 \cdot 10^{-12}$	4.755	$-9.6 \cdot 10^{+04}$	$5.9 \cdot 10^{-08}$	1.998	$3.8 \cdot 10^{-01}$
$1/5 \cdot 2^{-10}$	$2.2 \cdot 10^{-14}$	-2.741	$1.5 \cdot 10^{-24}$	$1.4 \cdot 10^{-09}$	1.999	$3.8 \cdot 10^{-01}$
$1/5 \cdot 2^{-11}$	$1.4 \cdot 10^{-13}$	-1.316	$7.7 \cdot 10^{-18}$	$3.7 \cdot 10^{-09}$	2.006	$4.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-12}$	$3.6 \cdot 10^{-12}$	-0.769	$-1.7 \cdot 10^{-16}$	$9.2 \cdot 10^{-10}$	2.033	$5.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-13}$	$6.2 \cdot 10^{-12}$	-1.224	$-1.4 \cdot 10^{-18}$	$2.2 \cdot 10^{-11}$	1.633	$7.7 \cdot 10^{-03}$
$1/5 \cdot 2^{-14}$	$1.4 \cdot 10^{-12}$	-0.773	$2.3 \cdot 10^{-16}$	$7.2 \cdot 10^{-11}$	0.873	$1.4 \cdot 10^{-07}$
$1/5 \cdot 2^{-15}$	$2.5 \cdot 10^{-12}$	-1.226	$1.0 \cdot 10^{-18}$	$3.9 \cdot 10^{-11}$	-0.479	$1.2 \cdot 10^{-14}$

Table 4: Convergence of the Adams-Bashforth method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0.00625$.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5$	$2.1 \cdot 10^{-03}$	14.109	$-1.5 \cdot 10^{+07}$	$1.4 \cdot 10^{-01}$	7.507	$2.6 \cdot 10^{+04}$
$1/5 \cdot 2^{-1}$	$1.2 \cdot 10^{-07}$	28.498	$3.7 \cdot 10^{+22}$	$8.1 \cdot 10^{-03}$	9.815	$5.3 \cdot 10^{+07}$
$1/5 \cdot 2^{-2}$	$3.1 \cdot 10^{-15}$	-0.934	$1.9 \cdot 10^{-17}$	$9.0 \cdot 10^{-06}$	4.982	$-2.7 \cdot 10^{+00}$
$1/5 \cdot 2^{-3}$	$6.0 \cdot 10^{-15}$	-1.266	$5.6 \cdot 10^{-17}$	$2.8 \cdot 10^{-08}$	4.995	$-2.8 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$1.4 \cdot 10^{-15}$	-0.782	$-4.7 \cdot 10^{-16}$	$8.9 \cdot 10^{-09}$	4.998	$-2.9 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$2.5 \cdot 10^{-15}$	-1.209	$-5.4 \cdot 10^{-17}$	$2.8 \cdot 10^{-11}$	5.009	$-3.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$5.8 \cdot 10^{-14}$	-0.785	$6.2 \cdot 10^{-16}$	$8.7 \cdot 10^{-12}$	5.645	$-1.2 \cdot 10^{+02}$
$1/5 \cdot 2^{-7}$	$1.0 \cdot 10^{-14}$	-1.225	$3.6 \cdot 10^{-17}$	$1.7 \cdot 10^{-14}$	-0.477	$-7.9 \cdot 10^{-15}$
$1/5 \cdot 2^{-8}$	$2.3 \cdot 10^{-14}$	-0.777	$-8.9 \cdot 10^{-16}$	$2.4 \cdot 10^{-14}$	-0.726	$-1.3 \cdot 10^{-16}$
$1/5 \cdot 2^{-9}$	$4.0 \cdot 10^{-13}$	-1.221	$-2.7 \cdot 10^{-18}$	$4.0 \cdot 10^{-13}$	-1.220	$-2.7 \cdot 10^{-18}$
$1/5 \cdot 2^{-10}$	$9.3 \cdot 10^{-13}$	-0.778	$1.2 \cdot 10^{-16}$	$9.3 \cdot 10^{-13}$	-0.778	$1.2 \cdot 10^{-16}$
$1/5 \cdot 2^{-11}$	$1.6 \cdot 10^{-13}$	-1.222	$2.0 \cdot 10^{-18}$	$1.6 \cdot 10^{-13}$	-1.222	$2.0 \cdot 10^{-18}$
$1/5 \cdot 2^{-12}$	$3.7 \cdot 10^{-12}$	-0.777	$-1.6 \cdot 10^{-16}$	$3.7 \cdot 10^{-12}$	-0.777	$-1.6 \cdot 10^{-16}$
$1/5 \cdot 2^{-13}$	$6.4 \cdot 10^{-12}$	-1.222	$-1.4 \cdot 10^{-18}$	$6.4 \cdot 10^{-12}$	-1.222	$-1.4 \cdot 10^{-18}$
$1/5 \cdot 2^{-14}$	$1.4 \cdot 10^{-12}$	-0.777	$2.2 \cdot 10^{-16}$	$1.4 \cdot 10^{-12}$	-0.777	$2.2 \cdot 10^{-16}$
$1/5 \cdot 2^{-15}$	$2.5 \cdot 10^{-12}$	-1.222	$1.0 \cdot 10^{-18}$	$2.5 \cdot 10^{-12}$	-1.222	$1.0 \cdot 10^{-18}$

Table 5: Convergence of the Adams-Bashforth method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0$.

4.3 Example 2.2: $\lambda_2 > \lambda_1 > 0$

We investigate

$$y''(t) = \frac{\lambda_1 + \lambda_2 - 1}{t} y'(t) - \frac{\lambda_1 \lambda_2}{t^2} y(t) + f(t), \quad 0 < t \leq 1,$$
$$y(1) = 2 + e, \quad y'(1) = \lambda_1 + \lambda_2 + 5e.$$

Its exact solution is $y(t) = t^4 e^t + t^{\lambda_1} + t^{\lambda_2}$, where $\lambda_1 = 1$ and $\lambda_2 = 3, 5$.

Tables 6 to 8 show the absolute errors and convergence orders for the Adams-Moulton formula. Both the expected order of convergence, and the stabilization of the error constants have been observed.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5 \cdot 2^{-1}$	$7.8 \cdot 10^{-02}$	2.000	$-7.8 \cdot 10^{+00}$	$5.2 \cdot 10^{-02}$	1.997	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-2}$	$1.9 \cdot 10^{-03}$	2.000	$-7.8 \cdot 10^{+00}$	$1.3 \cdot 10^{-03}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-3}$	$4.8 \cdot 10^{-03}$	2.000	$-7.8 \cdot 10^{+00}$	$3.2 \cdot 10^{-03}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$1.2 \cdot 10^{-04}$	2.000	$-7.8 \cdot 10^{+00}$	$8.2 \cdot 10^{-04}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$3.0 \cdot 10^{-05}$	2.000	$-7.8 \cdot 10^{+00}$	$2.0 \cdot 10^{-05}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$7.6 \cdot 10^{-05}$	2.000	$-7.8 \cdot 10^{+00}$	$5.1 \cdot 10^{-05}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$1.9 \cdot 10^{-06}$	2.000	$-7.8 \cdot 10^{+00}$	$1.2 \cdot 10^{-06}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-8}$	$4.7 \cdot 10^{-06}$	1.999	$-7.8 \cdot 10^{+00}$	$3.2 \cdot 10^{-06}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-9}$	$1.1 \cdot 10^{-07}$	2.000	$-7.8 \cdot 10^{+00}$	$8.0 \cdot 10^{-07}$	2.000	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-10}$	$2.9 \cdot 10^{-08}$	2.000	$-7.8 \cdot 10^{+00}$	$2.0 \cdot 10^{-08}$	2.000	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-11}$	$7.4 \cdot 10^{-08}$	1.999	$-7.8 \cdot 10^{+00}$	$5.0 \cdot 10^{-08}$	1.999	$-5.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-12}$	$1.8 \cdot 10^{-09}$	1.998	$-7.7 \cdot 10^{+00}$	$1.2 \cdot 10^{-09}$	1.998	$-5.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-13}$	$4.6 \cdot 10^{-09}$	2.022	$-9.9 \cdot 10^{+00}$	$3.1 \cdot 10^{-10}$	2.019	$-6.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-14}$	$1.1 \cdot 10^{-10}$	2.100	$-2.3 \cdot 10^{+00}$	$7.7 \cdot 10^{-10}$	2.107	$-1.7 \cdot 10^{+00}$
$1/5 \cdot 2^{-15}$	$2.6 \cdot 10^{-11}$	0.934	$-1.9 \cdot 10^{-06}$	$1.7 \cdot 10^{-11}$	1.056	$-5.8 \cdot 10^{-05}$

Table 6: Convergence of the Adams-Moulton method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0.1$.

This example is representative for all models with eigenvalues $\lambda_1, \lambda_2 \geq 0$ and $h^* > 0$.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5 \cdot 2^{-3}$	$1.2 \cdot 10^{-04}$	2.000	$-2.0 \cdot 10^{-01}$	$8.3 \cdot 10^{-04}$	1.999	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-4}$	$3.2 \cdot 10^{-04}$	2.000	$-2.0 \cdot 10^{-01}$	$2.0 \cdot 10^{-05}$	1.999	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$8.0 \cdot 10^{-05}$	2.000	$-2.0 \cdot 10^{-01}$	$5.2 \cdot 10^{-05}$	1.999	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$2.0 \cdot 10^{-06}$	2.000	$-2.0 \cdot 10^{-01}$	$1.3 \cdot 10^{-06}$	1.999	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$5.0 \cdot 10^{-06}$	2.000	$-2.0 \cdot 10^{-01}$	$3.2 \cdot 10^{-06}$	1.999	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$1.2 \cdot 10^{-07}$	1.999	$-2.0 \cdot 10^{-01}$	$8.1 \cdot 10^{-07}$	1.999	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$3.1 \cdot 10^{-08}$	2.000	$-2.0 \cdot 10^{-01}$	$2.0 \cdot 10^{-08}$	2.000	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-10}$	$7.8 \cdot 10^{-08}$	2.000	$-2.0 \cdot 10^{-01}$	$5.1 \cdot 10^{-08}$	2.000	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-11}$	$1.9 \cdot 10^{-09}$	1.998	$-2.0 \cdot 10^{-01}$	$1.2 \cdot 10^{-09}$	1.998	$-1.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-12}$	$4.8 \cdot 10^{-09}$	1.993	$-1.9 \cdot 10^{-01}$	$3.2 \cdot 10^{-09}$	1.991	$-1.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-13}$	$1.2 \cdot 10^{-10}$	2.081	$-4.9 \cdot 10^{+00}$	$8.0 \cdot 10^{-10}$	2.111	$-4.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-14}$	$2.9 \cdot 10^{-11}$	2.499	$-5.5 \cdot 10^{+02}$	$1.8 \cdot 10^{-11}$	2.805	$-1.1 \cdot 10^{+03}$
$1/5 \cdot 2^{-15}$	$5.1 \cdot 10^{-11}$	−0.653	$-2.0 \cdot 10^{-15}$	$2.6 \cdot 10^{-12}$	−1.321	$-3.4 \cdot 10^{-18}$

Table 7: Convergence of the Adams-Moulton method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0.025$.

h	$\delta_{(1,3)}$	$p_{(1,3)}$	$c_{(1,3)}$	$\delta_{(1,5)}$	$p_{(1,5)}$	$c_{(1,5)}$
$1/5 \cdot 2^{-5}$	$2.0 \cdot 10^{-06}$	2.000	$-5.1 \cdot 10^{-01}$	$1.3 \cdot 10^{-06}$	1.999	$-3.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$5.0 \cdot 10^{-06}$	2.000	$-5.1 \cdot 10^{-01}$	$3.2 \cdot 10^{-06}$	1.999	$-3.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$1.2 \cdot 10^{-07}$	1.999	$-5.1 \cdot 10^{-01}$	$8.2 \cdot 10^{-07}$	1.999	$-3.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$3.1 \cdot 10^{-08}$	1.999	$-5.1 \cdot 10^{-01}$	$2.0 \cdot 10^{-08}$	1.999	$-3.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$7.8 \cdot 10^{-08}$	2.000	$-5.1 \cdot 10^{-01}$	$5.1 \cdot 10^{-08}$	2.000	$-3.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-10}$	$1.9 \cdot 10^{-09}$	2.000	$-5.1 \cdot 10^{-01}$	$1.2 \cdot 10^{-09}$	2.000	$-3.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-11}$	$4.9 \cdot 10^{-09}$	1.995	$-4.9 \cdot 10^{-01}$	$3.2 \cdot 10^{-09}$	1.992	$-3.1 \cdot 10^{-02}$
$1/5 \cdot 2^{-12}$	$1.2 \cdot 10^{-10}$	1.974	$-4.0 \cdot 10^{-01}$	$8.0 \cdot 10^{-10}$	1.962	$-2.3 \cdot 10^{-02}$
$1/5 \cdot 2^{-13}$	$3.1 \cdot 10^{-11}$	2.344	$-2.0 \cdot 10^{+00}$	$2.0 \cdot 10^{-11}$	2.542	$-1.0 \cdot 10^{+01}$
$1/5 \cdot 2^{-14}$	$6.1 \cdot 10^{-11}$	3.294	$-9.5 \cdot 10^{+05}$	$3.5 \cdot 10^{-11}$	1.498	$-8.1 \cdot 10^{-04}$
$1/5 \cdot 2^{-15}$	$6.2 \cdot 10^{-12}$	−3.370	$1.6 \cdot 10^{-30}$	$1.2 \cdot 10^{-12}$	−2.294	$1.3 \cdot 10^{-24}$

Table 8: Convergence of the Adams-Moulton method; $\lambda_1 = 1,\ \lambda_2 = 3, 5;\ h^* = 0.00625$.

4.4 Example 3: $\lambda = \lambda_1 = \lambda_2 < 0$

We study now the model problem

$$y''(t) = \frac{2\lambda - 1}{t} y'(t) - \frac{\lambda^2}{t^2} y(t) + f(t), \quad 0 < t \leq 1,$$
$$y(0) = 0, \quad y'(0) = 0,$$

where $\lambda = \lambda_1 = \lambda_2 = -1, -3, -5, -7, -10$. In this case the only bounded solution of the homogeneous problem is zero and consequently, the solution $y(t)$ is a particular solution of the inhomogeneous problem. For appropriate choices of $f(t)$ we have $y(t) = t^2 e^t, t^4, t^4 e^t, t^7, t^7 e^t, t^{12}$. Note that we have ordered the sequence after the increasing order of $t = 0$ as a zero of $y(t)$.

Table 9 lists the observed convergence orders p of the Adams-Moulton second order formula for varying λ and $y(t)$. It turns out that the multiplicity of the singular point $t = 0$ as zero of the solution $y(t)$ does not influence the convergence properties of the method in this range of λ.

λ	-1	-3	-5	-7	-10
$y(t)$	p				
$t^2 e^t$	1.999980	1.999981	1.999723	1.999603	1.999207
t^4	1.999982	1.999921	1.999801	1.999621	1.999233
$t^4 e^t$	1.999991	1.999974	1.999942	1.999913	1.999841
t^7	1.999993	1.999986	1.999975	1.999962	1.999936
$t^7 e^t$	1.999994	1.999990	1.999984	1.999976	1.999961
t^{12}	1.999997	1.999995	1.999993	1.999991	1.999987

Table 9: Convergence of the Adams-Moulton method of second order.

4.5 Example 4

Here we compare two test problems

$$y''(t) = \frac{2\lambda - 1}{t} y'(t) - \frac{\lambda^2}{t^2} y(t) + f(t), \quad 0 < t \leq 1,$$
$$y(0) = 0, \quad y'(0) = 0,$$

where $\lambda = \lambda_1 = \lambda_2 = -1, -3, -5, -7, -10$ and $y(t) = t^7, t^{12}$ and

$$y''(t) = \frac{2\lambda - 1}{t} y'(t) - \frac{\lambda^2}{t^2} y(t), \quad 0 < t \leq 1,$$
$$y(1) = 1, \quad y'(1) = 1 + \lambda,$$

where $\lambda = \lambda_1 = \lambda_2 = 8, 13$ and $y(t) = (1 + \ln(t))t^\lambda$.

These problems show following similarities:

- In both cases λ is a twofold eigenvalue of M.
- The integration is carried out in the direction the solution decays.
- The solution pairs $y(t) = t^7$, $y(t) = (1 + \ln(t))t^8$ and $y(t) = t^{12}$, $y(t) = (1 + \ln(t))t^{13}$ have the same properties concerning smoothness and the order of $t = 0$ as a zero of $y(t)$.

Nevertheless, the problems are handled in a different manner, see Table 10, where the performance of the Adams-Bashforth method is illustrated. For the integration towards $t = 0$, $h^* = 0$.

h	$y(t)=t^7$			$y(t)=(1+\ln(t))t^8$		
	$\delta_{(-1,-1)}$	$p_{(-1,-1)}$	$c_{(-1,-1)}$	$\delta_{(8,8)}$	$p_{(8,8)}$	$c_{(8,8)}$
$1/5$	$1.8\cdot10^{-01}$	1.509	$2.0\cdot10^{+00}$	$7.7\cdot10^{+04}$	-4.934	$-2.7\cdot10^{+00}$
$1/5\cdot2^{-1}$	$6.4\cdot10^{-01}$	1.717	$3.3\cdot10^{+01}$	$2.3\cdot10^{+05}$	2.995	$2.3\cdot10^{+08}$
$1/5\cdot2^{-2}$	$1.9\cdot10^{-02}$	1.850	$5.0\cdot10^{+01}$	$2.9\cdot10^{+04}$	32.011	$1.3\cdot10^{+46}$
$1/5\cdot2^{-3}$	$5.4\cdot10^{-02}$	1.923	$6.5\cdot10^{+01}$	$6.8\cdot10^{-05}$	32.603	$1.1\cdot10^{+47}$
$1/5\cdot2^{-4}$	$1.4\cdot10^{-03}$	1.961	$7.7\cdot10^{+01}$	$1.0\cdot10^{-15}$	5.219	$-8.9\cdot10^{-05}$
$1/5\cdot2^{-5}$	$3.6\cdot10^{-03}$	1.980	$8.5\cdot10^{+01}$	$2.8\cdot10^{-17}$	6.849	$-3.5\cdot10^{-01}$
$1/5\cdot2^{-6}$	$9.3\cdot10^{-04}$	1.990	$9.0\cdot10^{+01}$	$2.4\cdot10^{-19}$	7.489	$-1.4\cdot10^{+00}$
$1/5\cdot2^{-7}$	$2.3\cdot10^{-05}$	1.995	$9.3\cdot10^{+01}$	$1.3\cdot10^{-21}$	7.380	$-6.9\cdot10^{+00}$
$1/5\cdot2^{-8}$	$5.9\cdot10^{-05}$	1.997	$9.5\cdot10^{+01}$	$8.1\cdot10^{-23}$	11.620	$1.0\cdot10^{+13}$
$1/5\cdot2^{-9}$	$1.4\cdot10^{-06}$	1.998	$9.6\cdot10^{+01}$	$2.5\cdot10^{-27}$	7.578	$-1.7\cdot10^{-01}$
$1/5\cdot2^{-10}$	$3.7\cdot10^{-06}$	1.999	$9.6\cdot10^{+01}$	$1.3\cdot10^{-29}$	9.671	$1.0\cdot10^{+07}$
$1/5\cdot2^{-11}$	$9.2\cdot10^{-07}$	1.999	$9.6\cdot10^{+01}$	$1.6\cdot10^{-32}$	4.416	$8.5\cdot10^{-14}$
$1/5\cdot2^{-12}$	$2.3\cdot10^{-08}$	1.999	$9.6\cdot10^{+01}$	$7.7\cdot10^{-33}$	7.170	$6.3\cdot10^{-02}$
$1/5\cdot2^{-13}$	$5.7\cdot10^{-08}$	2.005	$1.0\cdot10^{+01}$	$5.3\cdot10^{-35}$	9.403	$1.2\cdot10^{+08}$
$1/5\cdot2^{-14}$	$1.4\cdot10^{-09}$	2.043	$1.5\cdot10^{+01}$	$7.9\cdot10^{-38}$	5.734	$1.1\cdot10^{-10}$
$1/5\cdot2^{-15}$	$3.5\cdot10^{-09}$	1.683	$2.0\cdot10^{-01}$	$1.4\cdot10^{-40}$	9.307	$5.1\cdot10^{+09}$

h	$y(t)=t^{12}$			$y(t)=(1+\ln(t))t^{13}$		
	$\delta_{(-1,-1)}$	$p_{(-1,-1)}$	$c_{(-1,-1)}$	$\delta_{(13,13)}$	$p_{(13,13)}$	$c_{(13,13)}$
$1/5$	$4.6\cdot10^{+00}$	1.242	$3.4\cdot10^{+01}$	$1.5\cdot10^{+05}$	-11.534	$-1.3\cdot10^{-03}$
$1/5\cdot2^{-1}$	$1.9\cdot10^{-01}$	1.512	$6.4\cdot10^{+01}$	$4.6\cdot10^{+09}$	-9.3574	$2.0\cdot10^{-01}$
$1/5\cdot2^{-2}$	$6.8\cdot10^{-01}$	1.726	$1.2\cdot10^{+01}$	$3.0\cdot10^{+11}$	10.607	$1.9\cdot10^{+25}$
$1/5\cdot2^{-3}$	$2.0\cdot10^{-02}$	1.855	$1.9\cdot10^{+01}$	$1.9\cdot10^{+08}$	57.556	$3.1\cdot10^{+101}$
$1/5\cdot2^{-4}$	$5.7\cdot10^{-02}$	1.925	$2.6\cdot10^{+01}$	$9.2\cdot10^{-09}$	11.863	$3.4\cdot10^{+14}$
$1/5\cdot2^{-5}$	$1.5\cdot10^{-03}$	1.962	$3.2\cdot10^{+02}$	$2.4\cdot10^{-13}$	11.094	$7.0\cdot10^{+12}$
$1/5\cdot2^{-6}$	$3.8\cdot10^{-03}$	1.981	$3.5\cdot10^{+02}$	$1.1\cdot10^{-16}$	16.723	$8.8\cdot10^{+26}$
$1/5\cdot2^{-7}$	$9.8\cdot10^{-04}$	1.990	$3.7\cdot10^{+02}$	$1.0\cdot10^{-21}$	12.247	$2.4\cdot10^{+13}$
$1/5\cdot2^{-8}$	$2.4\cdot10^{-05}$	1.995	$3.9\cdot10^{+02}$	$2.1\cdot10^{-25}$	12.676	$-5.2\cdot10^{+15}$
$1/5\cdot2^{-9}$	$6.2\cdot10^{-05}$	1.997	$4.0\cdot10^{+02}$	$3.2\cdot10^{-28}$	10.896	$-4.5\cdot10^{+09}$
$1/5\cdot2^{-10}$	$1.5\cdot10^{-06}$	1.998	$4.0\cdot10^{+02}$	$1.7\cdot10^{-32}$	12.072	$-1.0\cdot10^{+13}$
$1/5\cdot2^{-11}$	$3.8\cdot10^{-06}$	1.999	$4.0\cdot10^{+02}$	$4.0\cdot10^{-35}$	15.071	$1.1\cdot10^{+25}$
$1/5\cdot2^{-12}$	$9.7\cdot10^{-07}$	1.999	$4.0\cdot10^{+02}$	$1.1\cdot10^{-40}$	13.225	$1.2\cdot10^{+17}$
$1/5\cdot2^{-13}$	$2.4\cdot10^{-08}$	2.002	$4.2\cdot10^{+02}$	$1.2\cdot10^{-44}$	11.627	$-5.1\cdot10^{+10}$
$1/5\cdot2^{-14}$	$6.0\cdot10^{-08}$	2.020	$5.1\cdot10^{+02}$	$3.8\cdot10^{-47}$	13.053	$-5.2\cdot10^{+17}$
$1/5\cdot2^{-15}$	$1.4\cdot10^{-09}$	1.811	$4.2\cdot10^{+01}$	$4.5\cdot10^{-51}$	11.957	$-1.0\cdot10^{+11}$

Table 10: The performance of the Adams-Bashforth formula of second order.

h	$\delta_{(8,8)}$	$p_{(8,8)}$	$c_{(8,8)}$	$\delta_{(13,13)}$	$p_{(13,13)}$	$c_{(13,13)}$
$1/5 \cdot 2^{-1}$	$3.6 \cdot 10^{+04}$	7.405	$-9.2 \cdot 10^{+11}$	$3.1 \cdot 10^{+06}$	-3.842	$-4.4 \cdot 10^{+03}$
$1/5 \cdot 2^{-2}$	$2.1 \cdot 10^{+01}$	30.547	$1.1 \cdot 10^{+41}$	$4.4 \cdot 10^{+08}$	19.122	$3.3 \cdot 10^{+33}$
$1/5 \cdot 2^{-3}$	$1.3 \cdot 10^{-08}$	1.503	$3.5 \cdot 10^{-05}$	$7.8 \cdot 10^{+02}$	44.173	$4.6 \cdot 10^{+73}$
$1/5 \cdot 2^{-4}$	$4.8 \cdot 10^{-08}$	1.771	$1.1 \cdot 10^{-05}$	$3.9 \cdot 10^{-11}$	3.732	$-5.0 \cdot 10^{-04}$
$1/5 \cdot 2^{-5}$	$1.4 \cdot 10^{-09}$	1.964	$3.0 \cdot 10^{-05}$	$2.9 \cdot 10^{-13}$	3.484	$-1.4 \cdot 10^{-05}$
$1/5 \cdot 2^{-6}$	$3.6 \cdot 10^{-09}$	2.006	$3.8 \cdot 10^{-04}$	$2.6 \cdot 10^{-14}$	2.880	$-4.3 \cdot 10^{-06}$
$1/5 \cdot 2^{-7}$	$9.0 \cdot 10^{-10}$	2.010	$3.9 \cdot 10^{-04}$	$3.6 \cdot 10^{-14}$	2.334	$-1.2 \cdot 10^{-08}$
$1/5 \cdot 2^{-8}$	$2.2 \cdot 10^{-11}$	2.007	$3.8 \cdot 10^{-04}$	$7.1 \cdot 10^{-15}$	2.094	$-2.3 \cdot 10^{-09}$
$1/5 \cdot 2^{-9}$	$5.5 \cdot 10^{-11}$	2.003	$3.7 \cdot 10^{-04}$	$1.6 \cdot 10^{-16}$	2.023	$-1.3 \cdot 10^{-09}$
$1/5 \cdot 2^{-10}$	$1.3 \cdot 10^{-12}$	2.002	$3.7 \cdot 10^{-04}$	$4.1 \cdot 10^{-16}$	2.005	$-1.1 \cdot 10^{-09}$
$1/5 \cdot 2^{-11}$	$3.4 \cdot 10^{-12}$	2.001	$3.6 \cdot 10^{-04}$	$1.0 \cdot 10^{-17}$	2.001	$-1.0 \cdot 10^{-09}$
$1/5 \cdot 2^{-12}$	$8.6 \cdot 10^{-13}$	2.000	$3.6 \cdot 10^{-04}$	$2.5 \cdot 10^{-18}$	2.000	$-1.0 \cdot 10^{-09}$
$1/5 \cdot 2^{-13}$	$2.1 \cdot 10^{-14}$	2.000	$3.6 \cdot 10^{-04}$	$6.4 \cdot 10^{-18}$	1.999	$-1.0 \cdot 10^{-09}$
$1/5 \cdot 2^{-14}$	$5.4 \cdot 10^{-14}$	2.002	$3.7 \cdot 10^{-04}$	$1.6 \cdot 10^{-19}$	1.998	$-1.0 \cdot 10^{-09}$
$1/5 \cdot 2^{-15}$	$1.3 \cdot 10^{-15}$	1.970	$2.5 \cdot 10^{-05}$	$4.0 \cdot 10^{-19}$	2.016	$-1.3 \cdot 10^{-09}$

Table 11: The performance of the Adams-Bashforth formula of second order; $h^* = 0.1$.

h	$\delta_{(8,8)}$	$p_{(8,8)}$	$c_{(8,8)}$	$\delta_{(13,13)}$	$p_{(13,13)}$	$c_{(13,13)}$
$1/5 \cdot 2^{-3}$	$1.0 \cdot 10^{-07}$	17.955	$-6.1 \cdot 10^{+22}$	$1.3 \cdot 10^{+06}$	63.081	$-1.5 \cdot 10^{+107}$
$1/5 \cdot 2^{-4}$	$4.1 \cdot 10^{-12}$	-2.037	$5.4 \cdot 10^{-16}$	$1.3 \cdot 10^{-13}$	19.859	$8.3 \cdot 10^{+25}$
$1/5 \cdot 2^{-5}$	$1.6 \cdot 10^{-12}$	1.947	$-3.3 \cdot 10^{-07}$	$1.4 \cdot 10^{-19}$	5.815	$-9.3 \cdot 10^{-06}$
$1/5 \cdot 2^{-6}$	$4.3 \cdot 10^{-12}$	1.993	$-4.3 \cdot 10^{-07}$	$2.5 \cdot 10^{-21}$	-2.133	$1.1 \cdot 10^{-26}$
$1/5 \cdot 2^{-7}$	$1.1 \cdot 10^{-13}$	1.994	$-4.3 \cdot 10^{-07}$	$1.1 \cdot 10^{-20}$	1.707	$-6.8 \cdot 10^{-15}$
$1/5 \cdot 2^{-8}$	$2.7 \cdot 10^{-14}$	1.996	$-4.4 \cdot 10^{-07}$	$3.3 \cdot 10^{-20}$	1.948	$-3.8 \cdot 10^{-14}$
$1/5 \cdot 2^{-9}$	$6.9 \cdot 10^{-14}$	1.997	$-4.4 \cdot 10^{-07}$	$8.7 \cdot 10^{-21}$	1.992	$-5.4 \cdot 10^{-14}$
$1/5 \cdot 2^{-10}$	$1.7 \cdot 10^{-15}$	1.998	$-4.5 \cdot 10^{-07}$	$2.1 \cdot 10^{-22}$	2.000	$-5.8 \cdot 10^{-14}$
$1/5 \cdot 2^{-11}$	$4.3 \cdot 10^{-15}$	1.999	$-4.5 \cdot 10^{-07}$	$5.4 \cdot 10^{-22}$	2.001	$-5.8 \cdot 10^{-14}$
$1/5 \cdot 2^{-12}$	$1.0 \cdot 10^{-16}$	1.999	$-4.5 \cdot 10^{-07}$	$1.3 \cdot 10^{-23}$	2.001	$-5.8 \cdot 10^{-14}$
$1/5 \cdot 2^{-13}$	$2.7 \cdot 10^{-17}$	1.999	$-4.5 \cdot 10^{-07}$	$3.4 \cdot 10^{-23}$	2.000	$-5.7 \cdot 10^{-14}$
$1/5 \cdot 2^{-14}$	$6.7 \cdot 10^{-17}$	1.999	$-4.5 \cdot 10^{-07}$	$8.5 \cdot 10^{-24}$	2.000	$-5.7 \cdot 10^{-14}$
$1/5 \cdot 2^{-15}$	$1.6 \cdot 10^{-18}$	2.002	$-4.7 \cdot 10^{-07}$	$2.1 \cdot 10^{-25}$	2.000	$-5.7 \cdot 10^{-14}$

Table 12: The performance of the Adams-Bashforth formula of second order; $h^* = 0.025$.

h	$\delta_{(8,8)}$	$p_{(8,8)}$	$c_{(8,8)}$	$\delta_{(13,13)}$	$p_{(13,13)}$	$c_{(13,13)}$
$1/5 \cdot 2^{-5}$	$1.4 \cdot 10^{-15}$	4.338	$5.2 \cdot 10^{-05}$	$1.6 \cdot 10^{-15}$	16.610	$-6.7 \cdot 10^{+22}$
$1/5 \cdot 2^{-6}$	$7.1 \cdot 10^{-16}$	0.599	$-2.2 \cdot 10^{-15}$	$1.6 \cdot 10^{-20}$	21.538	$1.5 \cdot 10^{+34}$
$1/5 \cdot 2^{-7}$	$4.7 \cdot 10^{-16}$	1.709	$-2.9 \cdot 10^{-12}$	$5.4 \cdot 10^{-26}$	2.641	$1.4 \cdot 10^{-19}$
$1/5 \cdot 2^{-8}$	$1.4 \cdot 10^{-17}$	1.923	$-1.3 \cdot 10^{-11}$	$8.7 \cdot 10^{-27}$	1.872	$5.7 \cdot 10^{-21}$
$1/5 \cdot 2^{-9}$	$3.8 \cdot 10^{-17}$	1.979	$-2.1 \cdot 10^{-11}$	$2.3 \cdot 10^{-28}$	1.328	$-8.0 \cdot 10^{-23}$
$1/5 \cdot 2^{-10}$	$9.6 \cdot 10^{-18}$	1.994	$-2.4 \cdot 10^{-11}$	$9.5 \cdot 10^{-28}$	1.875	$-8.6 \cdot 10^{-21}$
$1/5 \cdot 2^{-11}$	$2.4 \cdot 10^{-19}$	1.998	$-2.5 \cdot 10^{-11}$	$2.5 \cdot 10^{-29}$	1.975	$-2.1 \cdot 10^{-21}$
$1/5 \cdot 2^{-12}$	$6.0 \cdot 10^{-19}$	1.999	$-2.5 \cdot 10^{-11}$	$6.5 \cdot 10^{-29}$	1.996	$-2.6 \cdot 10^{-21}$
$1/5 \cdot 2^{-13}$	$1.5 \cdot 10^{-20}$	1.999	$-2.5 \cdot 10^{-11}$	$1.6 \cdot 10^{-30}$	2.000	$-2.7 \cdot 10^{-21}$
$1/5 \cdot 2^{-14}$	$3.7 \cdot 10^{-20}$	1.999	$-2.5 \cdot 10^{-11}$	$4.1 \cdot 10^{-30}$	2.000	$-2.7 \cdot 10^{-21}$
$1/5 \cdot 2^{-15}$	$9.4 \cdot 10^{-21}$	2.000	$-2.5 \cdot 10^{-11}$	$1.0 \cdot 10^{-31}$	2.000	$-2.7 \cdot 10^{-21}$

Table 13: The performance of the Adams-Bashforth formula of second order; $h^* = 0.00625$.

5 Conclusion

The experimental results suggest the following conclusion:
The linear multistep methods show the usual classical behavior in the treatment of singular problems if all eigenvalues of M are nonpositive and the integration is therefore performed away from the singular point.

It is still a challenge to prove this result analytically, also for more general problem classes[11] and to answer the question the experiments suggest:

What is the cause of the nonstandard convergence behavior in case $\lambda_1 \geq \lambda_2 \geq 0$ with $h^* = 0$, while in case $h^* > 0$ classical order can be observed?

[11] Systems with constant and variable coefficients.

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