The Application of Multiple Shooting to Singular Boundary Value Problems

W. Auzinger, P. Kofler

Department of Applied Mathematics and Numerical Analysis,

University of Technology, Vienna, Austria

Contents

Singular BVPs

Goal: Application of **Multiple Shooting** to

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \quad t \in (0, 1],$$
$$g(z(0), z(1)) = 0$$

- $M(t) = M(0) + tC(t)$, $C \in C^p[0, 1]$, $p \geq 0$
- $\lambda(M(0))$: $Re(\lambda) < 0$ or $\lambda = 0$
- $f(t, z) \in C^p([0, 1] \times \mathbb{R}^n)$
- $\boxed{M(0)z(0) = 0}$ $\quad (\Longleftrightarrow z \in C[0, 1])$

$$\Downarrow$$

existence of unique solution $z \in C^{p+1}[0, 1]$

Why Multiple Shooting is used

- Deficiencies of other methods when used to solve singular ODEs
  - direct discretization of 2nd order BVPs
  - collocation methods
- Efficient standard method for solving BVPs
- Advantages of Multiple Shooting:
  Different strategies near the singularity for
  - error estimation
  - grid selection
- Problems of Multiple Shooting:
  Possible order reduction of high order methods when integrating singular IVPs

Underlying IVP-solver

- Runge-Kutta methods lose high order
- Remedy:
  - Choose a basic method of low order:
    implicit Euler method
  - Estimate the global discretization error:
    Zadunaisky's idea
    * solve "neighboring problem"
    * use known error as estimation for the unknown error of the original problem
    * improve approximation $O(h) \rightarrow O(h^2)$
  - Iteration ($\rightarrow$ acceleration technique):
    Iterated Defect Correction method

“Emden DE” – IDeC

$$\begin{cases} z'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(t) - t \begin{pmatrix} 0 \\ z_1^5(t) \end{pmatrix} \\ z(0) = (1,0)^T \end{cases}$$

h	$\|\varepsilon_h\|_\infty$	p	c	$\|\varepsilon_h^{(1)}\|_\infty$	$p^{(1)}$	$c^{(1)}$
$1/5$	$2.4{\cdot}10^{-02}$	0.834	$-9.2{\cdot}10^{-02}$	$7.7{\cdot}10^{-03}$	1.835	$-1.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-1}$	$1.3{\cdot}10^{-02}$	0.921	$-1.1{\cdot}10^{-01}$	$2.1{\cdot}10^{-03}$	1.918	$-1.7{\cdot}10^{-01}$
$1/5{\cdot}2^{-2}$	$7.1{\cdot}10^{-03}$	0.960	$-1.2{\cdot}10^{-01}$	$5.7{\cdot}10^{-04}$	1.959	$-2.0{\cdot}10^{-01}$
$1/5{\cdot}2^{-3}$	$3.6{\cdot}10^{-03}$	0.980	$-1.3{\cdot}10^{-01}$	$1.4{\cdot}10^{-04}$	1.979	$-2.1{\cdot}10^{-01}$
$1/5{\cdot}2^{-4}$	$1.8{\cdot}10^{-03}$	0.990	$-1.4{\cdot}10^{-01}$	$3.7{\cdot}10^{-05}$	1.989	$-2.2{\cdot}10^{-01}$
$1/5{\cdot}2^{-5}$	$9.3{\cdot}10^{-04}$	0.995	$-1.4{\cdot}10^{-01}$	$9.4{\cdot}10^{-06}$	1.994	$-2.3{\cdot}10^{-01}$
$1/5{\cdot}2^{-6}$	$4.7{\cdot}10^{-04}$	0.997	$-1.4{\cdot}10^{-01}$	$2.3{\cdot}10^{-06}$	1.997	$-2.3{\cdot}10^{-01}$
$1/5{\cdot}2^{-7}$	$2.3{\cdot}10^{-04}$	0.998	$-1.4{\cdot}10^{-01}$	$5.9{\cdot}10^{-07}$	1.998	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-8}$	$1.1{\cdot}10^{-04}$	0.999	$-1.5{\cdot}10^{-01}$	$1.4{\cdot}10^{-07}$	1.999	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-9}$	$5.8{\cdot}10^{-05}$	0.999	$-1.5{\cdot}10^{-01}$	$3.7{\cdot}10^{-08}$	1.999	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-10}$	$2.9{\cdot}10^{-05}$	0.999	$-1.5{\cdot}10^{-01}$	$9.2{\cdot}10^{-09}$	1.999	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-11}$	$1.4{\cdot}10^{-05}$	0.999	$-1.5{\cdot}10^{-01}$	$2.3{\cdot}10^{-09}$	2.000	$-2.4{\cdot}10^{-01}$

h	$\|\varepsilon_h^{(2)}\|_\infty$	$p^{(2)}$	$c^{(2)}$	$\|\varepsilon_h^{(3)}\|_\infty$	$p^{(3)}$	$c^{(3)}$
$1/5$	$1.6\cdot 10^{-03}$	2.924	$-1.8\cdot 10^{-01}$	$9.5\cdot 10^{-04}$	3.330	$-2.0\cdot 10^{-01}$
$1/5\cdot 2^{-1}$	$2.1\cdot 10^{-04}$	2.918	$-1.7\cdot 10^{-01}$	$9.5\cdot 10^{-05}$	3.525	$-3.1\cdot 10^{-01}$
$1/5\cdot 2^{-2}$	$2.8\cdot 10^{-05}$	2.731	$-1.0\cdot 10^{-01}$	$8.2\cdot 10^{-06}$	3.770	$-6.6\cdot 10^{-01}$
$1/5\cdot 2^{-3}$	$4.3\cdot 10^{-06}$	2.868	$-1.6\cdot 10^{-01}$	$6.0\cdot 10^{-07}$	3.889	$-1.0\cdot 10^{+00}$
$1/5\cdot 2^{-4}$	$5.9\cdot 10^{-07}$	2.934	$-2.2\cdot 10^{-01}$	$4.0\cdot 10^{-08}$	3.946	$-1.3\cdot 10^{+00}$
$1/5\cdot 2^{-5}$	$7.7\cdot 10^{-08}$	2.967	$-2.6\cdot 10^{-01}$	$2.6\cdot 10^{-09}$	3.973	$-1.5\cdot 10^{+00}$
$1/5\cdot 2^{-6}$	$9.8\cdot 10^{-09}$	2.983	$-2.9\cdot 10^{-01}$	$1.6\cdot 10^{-10}$	3.986	$-1.6\cdot 10^{+00}$
$1/5\cdot 2^{-7}$	$1.2\cdot 10^{-09}$	2.991	$-3.1\cdot 10^{-01}$	$1.0\cdot 10^{-11}$	3.993	$-1.7\cdot 10^{+00}$
$1/5\cdot 2^{-8}$	$1.5\cdot 10^{-10}$	2.995	$-3.1\cdot 10^{-01}$	$6.6\cdot 10^{-13}$	3.996	$-1.7\cdot 10^{+00}$

h	$\|\varepsilon_h^{(4)}\|_\infty$	$p^{(4)}$	$c^{(4)}$	$\|\varepsilon_h^{(5)}\|_\infty$	$p^{(5)}$	$c^{(5)}$
$1/5$	$8.1\cdot 10^{-04}$	4.954	$-2.3\cdot 10^{+00}$	$5.4\cdot 10^{-04}$	5.558	$-4.1\cdot 10^{+00}$
$1/5\cdot 2^{-1}$	$2.6\cdot 10^{-05}$	5.079	$-3.1\cdot 10^{+00}$	$1.1\cdot 10^{-05}$	4.921	$-9.5\cdot 10^{-01}$
$1/5\cdot 2^{-2}$	$7.7\cdot 10^{-07}$	5.095	$-3.3\cdot 10^{+00}$	$3.7\cdot 10^{-07}$	5.141	$-1.8\cdot 10^{+00}$
$1/5\cdot 2^{-3}$	$2.2\cdot 10^{-08}$	5.084	$-3.1\cdot 10^{+00}$	$1.0\cdot 10^{-08}$	5.102	$-1.6\cdot 10^{+00}$
$1/5\cdot 2^{-4}$	$6.6\cdot 10^{-10}$	4.997	$-2.1\cdot 10^{+00}$	$3.1\cdot 10^{-10}$	5.075	$-1.4\cdot 10^{+00}$
$1/5\cdot 2^{-5}$	$2.0\cdot 10^{-11}$	4.983	$-2.0\cdot 10^{+00}$	$9.2\cdot 10^{-12}$	5.041	$-1.1\cdot 10^{+00}$
$1/5\cdot 2^{-6}$	$6.6\cdot 10^{-13}$	4.991	$-2.1\cdot 10^{+00}$	$2.8\cdot 10^{-13}$	5.021	$-1.0\cdot 10^{+00}$
$1/5\cdot 2^{-7}$	$2.0\cdot 10^{-14}$	4.995	$-2.1\cdot 10^{+00}$	$8.6\cdot 10^{-15}$	5.010	$-9.9\cdot 10^{-01}$
$1/5\cdot 2^{-8}$	$6.5\cdot 10^{-16}$	4.997	$-2.2\cdot 10^{+00}$	$2.6\cdot 10^{-16}$	5.005	$-9.5\cdot 10^{-01}$

Single Shooting – linear case

Consider

$$z'(t) = \frac{M(t)}{t} z(t) + f(t), \quad t \in (0,1],$$

$$\begin{pmatrix} B_{a1} \\ B_{a2} \end{pmatrix} z(0) + \begin{pmatrix} 0 \\ B_{b2} \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ \beta_2 \end{pmatrix}$$

B_{a1} consists of $n-r$ linearly independent rows of $M(0)$ $\quad (B_{a1} z(0) = 0 \Longleftrightarrow M(0) z(0) = 0)$

- general solution by superposition is

  $z(t) = z_h(t) + z_p(t) = \overline{Y}(t)s + v(t)$

- reduced fundamental solution matrix $\overline{Y}(t)$,

$$\overline{Y}'(t) = \frac{M(t)}{t} \overline{Y}(t), \quad t \in (0,1],$$

$$B_{a1} \overline{Y}(0) = 0,$$

  consists of r fundamental modes

Single Shooting – linear case (2)

- necessary initial values are computed using the QR-decomposition of B_{a1}
- particular solution $v(t)$,

$$v'(t) = \frac{M(t)}{t} v(t) + f(t), \quad t \in (0,1],$$
$$v(0) = 0$$

- remaining r boundary conditions are used to determine s from

$$\overline{Q} s = \overline{\beta}$$

where

$$\begin{aligned} \overline{Q} &:= B_{a2}\overline{Y}(0) + B_{b2}\overline{Y}(1), \\ \overline{\beta} &:= \beta_2 - B_{b2} v(1) \end{aligned}$$

Single Shooting – nonlinear case

Consider

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \quad t \in (0,1],$$
$$B_{a1} z(0) = 0,$$
$$g(z(0), z(1)) = 0$$

- compute basis $\tilde{E}$ of kernel of $M(0)$ using QR-decomposition of B_{a1}
- use $z_s(0) := \tilde{E}\, s \in \ker(M(0))$, $s \in \mathbb{R}^r$
- use Newton's method to solve nonlinear

$$F(s) := g(z_s(0), z_s(1)) = 0$$

- Jacobian $F'(s) \in \mathbb{R}^{r \times r}$ instead of $\mathbb{R}^{n \times n}$

“Emden DE” – Single Shooting

$$\begin{cases} z'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(t) - t\begin{pmatrix} 0 \\ z_1^5(t) \end{pmatrix} \\ z_1(1) = \sqrt{3/4}, \quad z_2(0) = 0 \end{cases}$$

h	$\lvert\varepsilon_0\rvert$	p	c	$\lVert\varepsilon_h\rVert_\infty$	p	c
$1/5$	$7.6\cdot10^{-04}$	4.855	$-1.8\cdot10^{+00}$	$3.8\cdot10^{-04}$	3.708	$-1.5\cdot10^{-01}$
$1/5\cdot2^{-1}$	$2.6\cdot10^{-05}$	4.764	$-1.5\cdot10^{+00}$	$2.9\cdot10^{-05}$	4.434	$-8.1\cdot10^{-01}$
$1/5\cdot2^{-2}$	$9.7\cdot10^{-07}$	4.812	$-1.7\cdot10^{+00}$	$1.3\cdot10^{-06}$	4.746	$-2.0\cdot10^{+00}$
$1/5\cdot2^{-3}$	$3.4\cdot10^{-08}$	4.898	$-2.4\cdot10^{+00}$	$5.1\cdot10^{-08}$	4.862	$-3.1\cdot10^{+00}$
$1/5\cdot2^{-4}$	$1.1\cdot10^{-09}$	4.948	$-3.0\cdot10^{+00}$	$1.7\cdot10^{-09}$	4.935	$-4.3\cdot10^{+00}$
$1/5\cdot2^{-5}$	$3.7\cdot10^{-11}$	4.974	$-3.4\cdot10^{+00}$	$5.7\cdot10^{-11}$	4.966	$-5.1\cdot10^{+00}$
$1/5\cdot2^{-6}$	$1.1\cdot10^{-12}$	4.987	$-3.7\cdot10^{+00}$	$1.8\cdot10^{-12}$	4.983	$-5.6\cdot10^{+00}$
$1/5\cdot2^{-7}$	$3.7\cdot10^{-14}$	4.993	$-3.8\cdot10^{+00}$	$5.8\cdot10^{-14}$	4.991	$-5.9\cdot10^{+00}$
$1/5\cdot2^{-8}$	$1.1\cdot10^{-15}$	4.996	$-3.9\cdot10^{+00}$	$1.8\cdot10^{-15}$	4.995	$-6.1\cdot10^{+00}$
$1/5\cdot2^{-9}$	$3.7\cdot10^{-17}$	4.998	$-4.0\cdot10^{+00}$	$5.7\cdot10^{-17}$	4.997	$-6.2\cdot10^{+00}$
$1/5\cdot2^{-10}$	$1.1\cdot10^{-18}$	4.999	$-4.0\cdot10^{+00}$	$1.8\cdot10^{-18}$	4.998	$-6.2\cdot10^{+00}$
$1/5\cdot2^{-11}$	$3.6\cdot10^{-20}$	4.999	$-4.0\cdot10^{+00}$	$5.6\cdot10^{-20}$	4.998	$-6.2\cdot10^{+00}$

Multiple Shooting – linear case

- subdivide integration interval $[0,1]$,

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$$

- patched solution by superposition is

$$z_i(t) = \overline{Y_i}(t) s_i + v_i(t), \quad t \in [t_i, t_{i+1}]$$

- $[0, t_1]$ $(i = 0)$:
  - reduced fundamental matrix $\overline{Y_0}(t)$

    satisfies $B_{a1} \overline{Y_0}(0) = 0$
  - $s_0 \in \mathbb{R}^r$

- $[t_i, t_{i+1}], \quad i = 1, \ldots, N-1$:
  - full fundamental matrices $\overline{Y_i}(t)$

    satisfy $\overline{Y_i}(t_i) = I_n$
  - $s_i \in \mathbb{R}^n$

Multiple Shooting – linear case (2)

- particular solutions $v_i(t)$, $v_i(t_i) = 0$
- $s := (s_0, \dots, s_{N-1})$ is determined from

$$\overline{A}s = \overline{\beta}$$

representing

– continuity (or matching) conditions

$$z_i(t_{i+1}) = z_{i+1}(t_{i+1})$$

$$\overline{Y_i}(t_{i+1})s_i + v_i(t_{i+1}) = s_{i+1}$$

– boundary conditions

$$\beta_2 = B_{a2}z_0(0) + B_{b2}z_{N-1}(1)$$

$$\beta_2 = B_{a2}\overline{Y_0}(0)s_0 + B_{b2}\overline{Y}_{N-1}(1)s_{N-1} + \\ + B_{b2}v_{N-1}(1)$$

Multiple Shooting – nonlinear case

- solution $z(t)$ by patching

$$z(t) = z_i(t, s_i), \quad t \in [t_i, t_{i+1}],$$

where

$$z_i'(t) = \frac{M(t)}{t} z_i(t) + f(t, z_i(t))$$

with initial conditions

- $[0, t_1]$, $(i = 0)$:

  – compute basis $\tilde{E}$ of kernel of $M(0)$

  – $z_0(0, s_0) := \tilde{E}\, s_0 \in \ker(M(0))$

  – $s_0 \in \mathbb{R}^r$

- $[t_i, t_{i+1}], \quad i = 1, \ldots, N-1$:

  – $z_i(t_i, s_i) = s_i$

  – $s_i \in \mathbb{R}^n$

Multiple Shooting – nonlinear case (2)

- use Newton's method to solve nonlinear

$$F(s) = 0$$

with $s := (s_0, \ldots, s_{N-1})$ corresponding to

– continuity (or matching) conditions

$$z_i(t_{i+1}, s_i) = z_{i+1}(t_{i+1}, s_{i+1})$$

$$z_i(t_{i+1}, s_i) = s_{i+1}$$

– boundary conditions

$$0 = g(z_0(0, s_0), z_{N-1}(1, s_{N-1}))$$

“Emden DE” – Multiple Shooting

- N fixed (e.g. $N = 4$)
- h inside $[t_i, t_{i+1}]$ is decreased
- expected order $O(h^5)$ in regular case

h	$\lVert\varepsilon_s\rVert_\infty$	p	c	$\lVert\varepsilon_h\rVert_\infty$	p	c
$1/5{\cdot}2^{-2}$	$2.3{\cdot}10^{-06}$	4.736	$-3.4{\cdot}10^{+00}$	$2.3{\cdot}10^{-06}$	4.736	$-3.4{\cdot}10^{+00}$
$1/5{\cdot}2^{-3}$	$8.9{\cdot}10^{-08}$	4.806	$-4.4{\cdot}10^{+00}$	$8.9{\cdot}10^{-08}$	4.806	$-4.4{\cdot}10^{+00}$
$1/5{\cdot}2^{-4}$	$3.1{\cdot}10^{-09}$	4.883	$-6.2{\cdot}10^{+00}$	$3.1{\cdot}10^{-09}$	4.883	$-6.2{\cdot}10^{+00}$
$1/5{\cdot}2^{-5}$	$1.0{\cdot}10^{-10}$	4.936	$-8.2{\cdot}10^{+00}$	$1.0{\cdot}10^{-10}$	4.796	$-4.0{\cdot}10^{+00}$
$1/5{\cdot}2^{-6}$	$3.5{\cdot}10^{-12}$	4.967	$-9.8{\cdot}10^{+00}$	$3.8{\cdot}10^{-12}$	4.890	$-6.9{\cdot}10^{+00}$
$1/5{\cdot}2^{-7}$	$1.1{\cdot}10^{-13}$	4.983	$-1.0{\cdot}10^{+01}$	$1.3{\cdot}10^{-13}$	4.947	$-1.0{\cdot}10^{+01}$
$1/5{\cdot}2^{-8}$	$3.5{\cdot}10^{-15}$	4.991	$-1.1{\cdot}10^{+01}$	$4.2{\cdot}10^{-15}$	4.974	$-1.2{\cdot}10^{+01}$
$1/5{\cdot}2^{-9}$	$1.1{\cdot}10^{-16}$	4.995	$-1.1{\cdot}10^{+01}$	$1.3{\cdot}10^{-16}$	4.987	$-1.3{\cdot}10^{+01}$
$1/5{\cdot}2^{-10}$	$3.5{\cdot}10^{-18}$	4.997	$-1.2{\cdot}10^{+01}$	$4.2{\cdot}10^{-18}$	4.993	$-1.4{\cdot}10^{+01}$
$1/5{\cdot}2^{-11}$	$1.1{\cdot}10^{-19}$	4.998	$-1.2{\cdot}10^{+01}$	$1.3{\cdot}10^{-19}$	4.995	$-1.4{\cdot}10^{+01}$

“Emden DE” – Multiple Shooting (2)

- N is increased
- h proportional to $\frac{1}{N}$ such that
  – 1 step of IVP-solver inside $[t_i, t_{i+1}]$
- expected order $O(h^5)$ in regular case

h	$\|\varepsilon_s\|_\infty$	p	c	$\|\varepsilon_h\|_\infty$	p	c
$1/5\cdot 2^{-1}$	$5.1\cdot 10^{-05}$	4.445	$-1.4\cdot 10^{+00}$	$5.5\cdot 10^{-05}$	4.546	$-1.9\cdot 10^{+00}$
$1/5\cdot 2^{-2}$	$2.3\cdot 10^{-06}$	2.847	$-1.2\cdot 10^{-02}$	$2.3\cdot 10^{-06}$	2.806	$-1.0\cdot 10^{-02}$
$1/5\cdot 2^{-3}$	$3.3\cdot 10^{-07}$	1.652	$-1.4\cdot 10^{-04}$	$3.4\cdot 10^{-07}$	1.637	$-1.4\cdot 10^{-04}$
$1/5\cdot 2^{-4}$	$1.0\cdot 10^{-07}$	1.974	$-6.0\cdot 10^{-04}$	$1.0\cdot 10^{-07}$	1.971	$-6.1\cdot 10^{-04}$
$1/5\cdot 2^{-5}$	$2.6\cdot 10^{-08}$	2.023	$-7.7\cdot 10^{-04}$	$2.7\cdot 10^{-08}$	2.021	$-7.9\cdot 10^{-04}$
$1/5\cdot 2^{-6}$	$6.5\cdot 10^{-09}$	2.023	$-7.7\cdot 10^{-04}$	$6.8\cdot 10^{-09}$	2.022	$-8.0\cdot 10^{-04}$
$1/5\cdot 2^{-7}$	$1.6\cdot 10^{-09}$	2.015	$-7.3\cdot 10^{-04}$	$1.6\cdot 10^{-09}$	2.014	$-7.6\cdot 10^{-04}$
$1/5\cdot 2^{-8}$	$4.0\cdot 10^{-10}$	2.008	$-6.9\cdot 10^{-04}$	$4.1\cdot 10^{-10}$	2.008	$-7.2\cdot 10^{-04}$
$1/5\cdot 2^{-9}$	$9.9\cdot 10^{-11}$	2.004	$-6.7\cdot 10^{-04}$	$1.0\cdot 10^{-10}$	2.004	$-7.0\cdot 10^{-04}$
$1/5\cdot 2^{-10}$	$2.4\cdot 10^{-11}$	2.000	$-6.5\cdot 10^{-04}$	$2.5\cdot 10^{-11}$	2.000	$-6.8\cdot 10^{-04}$

“Emden DE” – Multiple Shooting (3)

- N is incresed **but** $t_1 := \frac{1}{8}$ fixed
- h proportional to $\frac{1}{N}$ such that
  - several steps of IVP-solver inside $[0, 1/8]$
  - 1 step of IVP-solver inside $[t_i, t_{i+1}]$
- expected order $O(h^5)$ in regular case

h	$\|\varepsilon_s\|_\infty$	p	c	$\|\varepsilon_h\|_\infty$	p	c
$1/5{\cdot}2^{-3}$	$3.3{\cdot}10^{-07}$	3.311	$-6.7{\cdot}10^{-02}$	$3.4{\cdot}10^{-07}$	3.351	$-7.9{\cdot}10^{-02}$
$1/5{\cdot}2^{-4}$	$3.3{\cdot}10^{-08}$	4.106	$-2.1{\cdot}10^{+00}$	$3.3{\cdot}10^{-08}$	4.100	$-2.1{\cdot}10^{+00}$
$1/5{\cdot}2^{-5}$	$1.9{\cdot}10^{-09}$	4.519	$-1.7{\cdot}10^{+01}$	$1.9{\cdot}10^{-09}$	4.525	$-1.8{\cdot}10^{+01}$
$1/5{\cdot}2^{-6}$	$8.4{\cdot}10^{-11}$	4.756	$-6.9{\cdot}10^{+01}$	$8.4{\cdot}10^{-11}$	4.757	$-7.0{\cdot}10^{+01}$
$1/5{\cdot}2^{-7}$	$3.1{\cdot}10^{-12}$	4.886	$-1.6{\cdot}10^{+02}$	$3.1{\cdot}10^{-12}$	4.886	$-1.6{\cdot}10^{+02}$
$1/5{\cdot}2^{-8}$	$1.8{\cdot}10^{-13}$	4.984	$-7.0{\cdot}10^{+02}$	$1.7{\cdot}10^{-13}$	4.984	$-6.5{\cdot}10^{+02}$
$1/5{\cdot}2^{-9}$	$5.6{\cdot}10^{-15}$	4.991	$-6.6{\cdot}10^{+02}$	$5.5{\cdot}10^{-15}$	4.995	$-6.3{\cdot}10^{+02}$
$1/5{\cdot}2^{-10}$	$1.7{\cdot}10^{-16}$	4.996	$-6.3{\cdot}10^{+02}$	$1.7{\cdot}10^{-16}$	4.998	$-6.2{\cdot}10^{+02}$
$1/5{\cdot}2^{-11}$	$5.4{\cdot}10^{-18}$	4.999	$-6.2{\cdot}10^{+02}$	$5.4{\cdot}10^{-18}$	4.999	$-6.1{\cdot}10^{+02}$
$1/5{\cdot}2^{-12}$	$1.6{\cdot}10^{-19}$	4.999	$-6.2{\cdot}10^{+02}$	$1.6{\cdot}10^{-19}$	4.999	$-6.1{\cdot}10^{+02}$

Conclusion

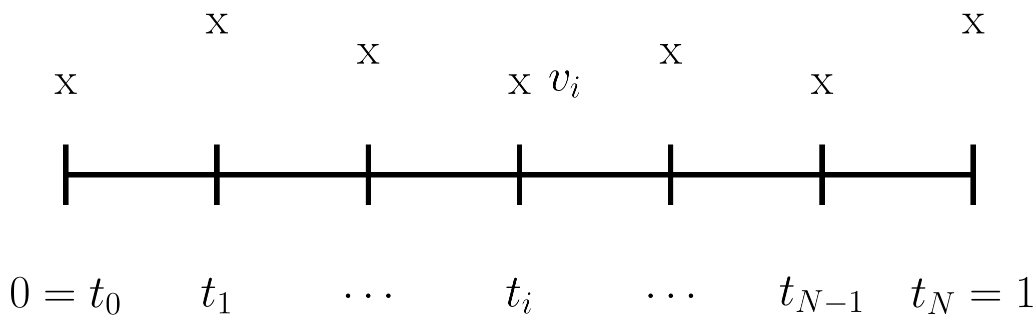
- IDeC is efficient
- Single Shooting is efficient
  if underlying IVP-integrator is efficient
  (Theory presented by O. Koch)
- Multiple Shooting works, but is not efficient

Further Investigations

- integrate from right to left to allow
  $\lambda(M(0)) : Re(\lambda) > 0$ or $\lambda = 0$
- idea how to couple both shooting methods
  to allow $\lambda(M(0)) : Re(\lambda) \neq 0$ or $\lambda = 0$

Implicit Euler method

- Runge-Kutta Methods lose their high convergence order.
- Remedy:
  - Choose a basic method of low order
  - Apply an acceleration technique



$$\Delta_h = (t_0, t_1, \ldots, t_i, \ldots, t_N),\ N = 1/h$$

$$v_h = (v_0, v_1, \ldots, v_i, \ldots, v_N)$$

$$\Downarrow$$

$$|v_i - z(t_i)| = O(h),\ i = 0(1)N,\ \text{for}\, h \to 0$$

Zadunaisky's idea

- solve

$$\begin{cases} z'(t) = F(t, z(t)),\ t \in (0,1] \\ z(0) = \beta \end{cases}$$

  by implicit Euler method

- obtain grid vector solution

  $z_h^{(0)} = (z_0^{(0)}, \ldots, z_i^{(0)}, \ldots, z_N^{(0)})$

- interpolate $(t_0, z_0^{(0)}), \ldots, (t_N, z_N^{(0)})$ by polynomial $p^{(0)}(t)$ of degree $m = N$

- solve "neighboring problem"

$$\begin{cases} z'(t) = F(t, z(t)) + p^{(0)\prime}(t) - F(t, p^{(0)}(t)) \\ z(0) = p^{(0)}(0) = \beta \end{cases}$$

Zadunaisky's idea (2)

- obtain grid vector solution $p_h^{(0)}$ by implicit Euler method on the same grid Δ_h
- use known global error $z_h^{(0)} - p_h^{(0)}$ as estimation for the unknown global error of the original problem
- accuracy by the heuristic argumentation

- implementation:
  - continuous function $p^{(0)}(t)$ composed of polynomials with degree $m < N$

Iterated Defect Correction (IDeC)

- improve approximation

$$z_h^{(1)} := z_h^{(0)} + (z_h^{(0)} - p_h^{(0)})$$

- interpolate $(t_0, z_0^{(1)}), \ldots, (t_N, z_N^{(1)})$ by new polynomial $p^{(1)}(t)$

- solve "neighboring problem" with defect

$$p^{(1)\prime}(t) - F(t, p^{(1)}(t))$$

- obtain grid vector solution $p_h^{(1)}$ by implicit Euler method on the same grid Δ_h

- improve approximation again

$$z_h^{(2)} := z_h^{(0)} + (z_h^{(1)} - p_h^{(1)})$$