

**The Application of  
Multiple Shooting to  
Singular Boundary Value Problems**

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## Singular BVPs

Goal: Application of **Multiple Shooting** to

$$z'(t) = \frac{M(t)}{t}z(t) + f(t, z(t)), \quad t \in (0, 1],$$

$$g(z(0), z(1)) = 0$$

- $M(t) = M(0) + tC(t)$ ,  $C \in C^p[0, 1]$ ,  $p \geq 0$
- $\lambda(M(0)) : \operatorname{Re}(\lambda) < 0$  or  $\lambda = 0$
- $f(t, z) \in C^p([0, 1] \times \mathbb{R}^n)$
- $\boxed{M(0)z(0) = 0}$  ( $\iff z \in C[0, 1]$ )

$\Downarrow$

existence of unique solution  $z \in C^{p+1}[0, 1]$

## Why Multiple Shooting is used

- Deficiencies of other methods when used to solve singular ODEs
  - direct discretization of 2<sup>nd</sup> order BVPs
  - collocation methods
- Efficient standard method for solving BVPs
- Advantages of Multiple Shooting:

Different strategies near the singularity for

  - error estimation
  - grid selection
- Problems of Multiple Shooting:

Possible order reduction of high order methods when integrating singular IVPs

## Underlying IVP-solver

- Runge-Kutta methods lose high order
- Remedy:
  - Choose a basic method of low order:  
implicit Euler method
  - Estimate the global discretization error:  
Zadunaisky's idea
    - \* solve “neighboring problem”
    - \* use known error as estimation for the unknown error of the original problem
    - \* improve approximation  $O(h) \rightarrow O(h^2)$
  - Iteration ( $\rightarrow$  acceleration technique):  
Iterated Defect Correction method

## “Emden DE” – IDeC

$$\begin{cases} z'(t) = \frac{1}{t} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(t) - t \begin{pmatrix} 0 \\ z_1^5(t) \end{pmatrix} \\ z(0) = (1, 0)^T \end{cases}$$

$h$	$\ \varepsilon_h\ _\infty$	$p$	$c$	$\ \varepsilon_h^{(1)}\ _\infty$	$p^{(1)}$	$c^{(1)}$
1/5	$2.4 \cdot 10^{-02}$	0.834	$-9.2 \cdot 10^{-02}$	$7.7 \cdot 10^{-03}$	1.835	$-1.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-1}$	$1.3 \cdot 10^{-02}$	0.921	$-1.1 \cdot 10^{-01}$	$2.1 \cdot 10^{-03}$	1.918	$-1.7 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$7.1 \cdot 10^{-03}$	0.960	$-1.2 \cdot 10^{-01}$	$5.7 \cdot 10^{-04}$	1.959	$-2.0 \cdot 10^{-01}$
$1/5 \cdot 2^{-3}$	$3.6 \cdot 10^{-03}$	0.980	$-1.3 \cdot 10^{-01}$	$1.4 \cdot 10^{-04}$	1.979	$-2.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-4}$	$1.8 \cdot 10^{-03}$	0.990	$-1.4 \cdot 10^{-01}$	$3.7 \cdot 10^{-05}$	1.989	$-2.2 \cdot 10^{-01}$
$1/5 \cdot 2^{-5}$	$9.3 \cdot 10^{-04}$	0.995	$-1.4 \cdot 10^{-01}$	$9.4 \cdot 10^{-06}$	1.994	$-2.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-6}$	$4.7 \cdot 10^{-04}$	0.997	$-1.4 \cdot 10^{-01}$	$2.3 \cdot 10^{-06}$	1.997	$-2.3 \cdot 10^{-01}$
$1/5 \cdot 2^{-7}$	$2.3 \cdot 10^{-04}$	0.998	$-1.4 \cdot 10^{-01}$	$5.9 \cdot 10^{-07}$	1.998	$-2.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$1.1 \cdot 10^{-04}$	0.999	$-1.5 \cdot 10^{-01}$	$1.4 \cdot 10^{-07}$	1.999	$-2.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-9}$	$5.8 \cdot 10^{-05}$	0.999	$-1.5 \cdot 10^{-01}$	$3.7 \cdot 10^{-08}$	1.999	$-2.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-10}$	$2.9 \cdot 10^{-05}$	0.999	$-1.5 \cdot 10^{-01}$	$9.2 \cdot 10^{-09}$	1.999	$-2.4 \cdot 10^{-01}$
$1/5 \cdot 2^{-11}$	$1.4 \cdot 10^{-05}$	0.999	$-1.5 \cdot 10^{-01}$	$2.3 \cdot 10^{-09}$	2.000	$-2.4 \cdot 10^{-01}$

$h$	$\ \varepsilon_h^{(2)}\ _\infty$	$p^{(2)}$	$c^{(2)}$	$\ \varepsilon_h^{(3)}\ _\infty$	$p^{(3)}$	$c^{(3)}$
$1/5$	$1.6 \cdot 10^{-03}$	2.924	$-1.8 \cdot 10^{-01}$	$9.5 \cdot 10^{-04}$	3.330	$-2.0 \cdot 10^{-01}$
$1/5 \cdot 2^{-1}$	$2.1 \cdot 10^{-04}$	2.918	$-1.7 \cdot 10^{-01}$	$9.5 \cdot 10^{-05}$	3.525	$-3.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$2.8 \cdot 10^{-05}$	2.731	$-1.0 \cdot 10^{-01}$	$8.2 \cdot 10^{-06}$	3.770	$-6.6 \cdot 10^{-01}$
$1/5 \cdot 2^{-3}$	$4.3 \cdot 10^{-06}$	2.868	$-1.6 \cdot 10^{-01}$	$6.0 \cdot 10^{-07}$	3.889	$-1.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$5.9 \cdot 10^{-07}$	2.934	$-2.2 \cdot 10^{-01}$	$4.0 \cdot 10^{-08}$	3.946	$-1.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$7.7 \cdot 10^{-08}$	2.967	$-2.6 \cdot 10^{-01}$	$2.6 \cdot 10^{-09}$	3.973	$-1.5 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$9.8 \cdot 10^{-09}$	2.983	$-2.9 \cdot 10^{-01}$	$1.6 \cdot 10^{-10}$	3.986	$-1.6 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$1.2 \cdot 10^{-09}$	2.991	$-3.1 \cdot 10^{-01}$	$1.0 \cdot 10^{-11}$	3.993	$-1.7 \cdot 10^{+00}$
$1/5 \cdot 2^{-8}$	$1.5 \cdot 10^{-10}$	2.995	$-3.1 \cdot 10^{-01}$	$6.6 \cdot 10^{-13}$	3.996	$-1.7 \cdot 10^{+00}$
$h$	$\ \varepsilon_h^{(4)}\ _\infty$	$p^{(4)}$	$c^{(4)}$	$\ \varepsilon_h^{(5)}\ _\infty$	$p^{(5)}$	$c^{(5)}$
$1/5$	$8.1 \cdot 10^{-04}$	4.954	$-2.3 \cdot 10^{+00}$	$5.4 \cdot 10^{-04}$	5.558	$-4.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-1}$	$2.6 \cdot 10^{-05}$	5.079	$-3.1 \cdot 10^{+00}$	$1.1 \cdot 10^{-05}$	4.921	$-9.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$7.7 \cdot 10^{-07}$	5.095	$-3.3 \cdot 10^{+00}$	$3.7 \cdot 10^{-07}$	5.141	$-1.8 \cdot 10^{+00}$
$1/5 \cdot 2^{-3}$	$2.2 \cdot 10^{-08}$	5.084	$-3.1 \cdot 10^{+00}$	$1.0 \cdot 10^{-08}$	5.102	$-1.6 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$6.6 \cdot 10^{-10}$	4.997	$-2.1 \cdot 10^{+00}$	$3.1 \cdot 10^{-10}$	5.075	$-1.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$2.0 \cdot 10^{-11}$	4.983	$-2.0 \cdot 10^{+00}$	$9.2 \cdot 10^{-12}$	5.041	$-1.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$6.6 \cdot 10^{-13}$	4.991	$-2.1 \cdot 10^{+00}$	$2.8 \cdot 10^{-13}$	5.021	$-1.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$2.0 \cdot 10^{-14}$	4.995	$-2.1 \cdot 10^{+00}$	$8.6 \cdot 10^{-15}$	5.010	$-9.9 \cdot 10^{-01}$
$1/5 \cdot 2^{-8}$	$6.5 \cdot 10^{-16}$	4.997	$-2.2 \cdot 10^{+00}$	$2.6 \cdot 10^{-16}$	5.005	$-9.5 \cdot 10^{-01}$

## Single Shooting – linear case

Consider

$$z'(t) = \frac{M(t)}{t}z(t) + f(t), \quad t \in (0, 1],$$
$$\begin{pmatrix} B_{a1} \\ B_{a2} \end{pmatrix} z(0) + \begin{pmatrix} 0 \\ B_{b2} \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ \beta_2 \end{pmatrix}$$

$B_{a1}$  consists of  $n - r$  linearly independent rows of  $M(0)$  ( $B_{a1}z(0) = 0 \iff M(0)z(0) = 0$ )

- general solution by superposition is

$$z(t) = z_h(t) + z_p(t) = \bar{Y}(t)s + v(t)$$

- reduced fundamental solution matrix  $\bar{Y}(t)$ ,

$$\bar{Y}'(t) = \frac{M(t)}{t}\bar{Y}(t), \quad t \in (0, 1],$$

$$B_{a1}\bar{Y}(0) = 0,$$

consists of  $r$  fundamental modes

## Single Shooting – linear case (2)

- necessary initial values are computed using the QR-decomposition of  $B_{a1}$

- particular solution  $v(t)$ ,

$$v'(t) = \frac{M(t)}{t}v(t) + f(t), \quad t \in (0, 1],$$

$$v(0) = 0$$

- remaining  $r$  boundary conditions are used to determine  $s$  from

$$\bar{Q}s = \bar{\beta}$$

where

$$\bar{Q} := B_{a2}\bar{Y}(0) + B_{b2}\bar{Y}(1),$$

$$\bar{\beta} := \beta_2 - B_{b2}v(1)$$

## Single Shooting – nonlinear case

Consider

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \quad t \in (0, 1],$$

$$B_{a1} z(0) = 0,$$

$$g(z(0), z(1)) = 0$$

- compute basis  $\tilde{E}$  of kernel of  $M(0)$  using QR-decomposition of  $B_{a1}$
- use  $z_s(0) := \tilde{E} s \in \ker(M(0))$ ,  $s \in \mathbb{R}^r$
- use Newton's method to solve nonlinear

$$F(s) := g(z_s(0), z_s(1)) = 0$$

- Jacobian  $F'(s) \in \mathbb{R}^{r \times r}$  instead of  $\mathbb{R}^{n \times n}$

## “Emden DE” – Single Shooting

$$\begin{cases} z'(t) = \frac{1}{t} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(t) - t \begin{pmatrix} 0 \\ z_1^5(t) \end{pmatrix} \\ z_1(1) = \sqrt{3/4}, \quad z_2(0) = 0 \end{cases}$$

$h$	$ \varepsilon_0 $	$p$	$c$	$\ \varepsilon_h\ _\infty$	$p$	$c$
1/5	$7.6 \cdot 10^{-04}$	4.855	$-1.8 \cdot 10^{+00}$	$3.8 \cdot 10^{-04}$	3.708	$-1.5 \cdot 10^{-01}$
$1/5 \cdot 2^{-1}$	$2.6 \cdot 10^{-05}$	4.764	$-1.5 \cdot 10^{+00}$	$2.9 \cdot 10^{-05}$	4.434	$-8.1 \cdot 10^{-01}$
$1/5 \cdot 2^{-2}$	$9.7 \cdot 10^{-07}$	4.812	$-1.7 \cdot 10^{+00}$	$1.3 \cdot 10^{-06}$	4.746	$-2.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-3}$	$3.4 \cdot 10^{-08}$	4.898	$-2.4 \cdot 10^{+00}$	$5.1 \cdot 10^{-08}$	4.862	$-3.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$1.1 \cdot 10^{-09}$	4.948	$-3.0 \cdot 10^{+00}$	$1.7 \cdot 10^{-09}$	4.935	$-4.3 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$3.7 \cdot 10^{-11}$	4.974	$-3.4 \cdot 10^{+00}$	$5.7 \cdot 10^{-11}$	4.966	$-5.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$1.1 \cdot 10^{-12}$	4.987	$-3.7 \cdot 10^{+00}$	$1.8 \cdot 10^{-12}$	4.983	$-5.6 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$3.7 \cdot 10^{-14}$	4.993	$-3.8 \cdot 10^{+00}$	$5.8 \cdot 10^{-14}$	4.991	$-5.9 \cdot 10^{+00}$
$1/5 \cdot 2^{-8}$	$1.1 \cdot 10^{-15}$	4.996	$-3.9 \cdot 10^{+00}$	$1.8 \cdot 10^{-15}$	4.995	$-6.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-9}$	$3.7 \cdot 10^{-17}$	4.998	$-4.0 \cdot 10^{+00}$	$5.7 \cdot 10^{-17}$	4.997	$-6.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-10}$	$1.1 \cdot 10^{-18}$	4.999	$-4.0 \cdot 10^{+00}$	$1.8 \cdot 10^{-18}$	4.998	$-6.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-11}$	$3.6 \cdot 10^{-20}$	4.999	$-4.0 \cdot 10^{+00}$	$5.6 \cdot 10^{-20}$	4.998	$-6.2 \cdot 10^{+00}$

## Multiple Shooting – linear case

- subdivide integration interval  $[0, 1]$ ,

$$0 = t_0 < t_1 < \dots < t_{N-1} < t_N = 1$$

- patched solution by superposition is

$$z_i(t) = \overline{Y}_i(t)s_i + v_i(t), \quad t \in [t_i, t_{i+1}]$$

- $[0, t_1]$  ( $i = 0$ ):

– reduced fundamental matrix  $\overline{Y}_0(t)$

satisfies  $B_{a1}\overline{Y}_0(0) = 0$

–  $s_0 \in \mathbb{R}^r$

- $[t_i, t_{i+1}]$ ,  $i = 1, \dots, N-1$ :

– full fundamental matrices  $\overline{Y}_i(t)$

satisfy  $\overline{Y}_i(t_i) = I_n$

–  $s_i \in \mathbb{R}^n$

## Multiple Shooting – linear case (2)

- particular solutions  $v_i(t)$ ,  $v_i(t_i) = 0$
- $s := (s_0, \dots, s_{N-1})$  is determined from

$$\bar{A}s = \bar{\beta}$$

representing

– continuity (or matching) conditions

$$z_i(t_{i+1}) = z_{i+1}(t_{i+1})$$

$$\bar{Y}_i(t_{i+1})s_i + v_i(t_{i+1}) = s_{i+1}$$

– boundary conditions

$$\beta_2 = B_{a2}z_0(0) + B_{b2}z_{N-1}(1)$$

$$\begin{aligned} \beta_2 &= B_{a2}\bar{Y}_0(0)s_0 + B_{b2}\bar{Y}_{N-1}(1)s_{N-1} + \\ &\quad + B_{b2}v_{N-1}(1) \end{aligned}$$

## Multiple Shooting – nonlinear case

- solution  $z(t)$  by patching

$$z(t) = z_i(t, s_i), \quad t \in [t_i, t_{i+1}],$$

where

$$z_i'(t) = \frac{M(t)}{t} z_i(t) + f(t, z_i(t))$$

with initial conditions

- $[0, t_1]$ , ( $i = 0$ ):
  - compute basis  $\tilde{E}$  of kernel of  $M(0)$
  - $z_0(0, s_0) := \tilde{E} s_0 \in \ker(M(0))$
  - $s_0 \in \mathbb{R}^r$
- $[t_i, t_{i+1}]$ ,  $i = 1, \dots, N-1$ :
  - $z_i(t_i, s_i) = s_i$
  - $s_i \in \mathbb{R}^n$

## Multiple Shooting – nonlinear case (2)

- use Newton's method to solve nonlinear

$$F(s) = 0$$

with  $s := (s_0, \dots, s_{N-1})$  corresponding to

– continuity (or matching) conditions

$$z_i(t_{i+1}, s_i) = z_{i+1}(t_{i+1}, s_{i+1})$$

$$z_i(t_{i+1}, s_i) = s_{i+1}$$

– boundary conditions

$$0 = g(z_0(0, s_0), z_{N-1}(1, s_{N-1}))$$

## “Emden DE” – Multiple Shooting

- $N$  fixed (e.g.  $N = 4$ )
- $h$  inside  $[t_i, t_{i+1}]$  is decreased
- expected order  $O(h^5)$  in regular case

$h$	$\ \varepsilon_s\ _\infty$	$p$	$c$	$\ \varepsilon_h\ _\infty$	$p$	$c$
$1/5 \cdot 2^{-2}$	$2.3 \cdot 10^{-06}$	4.736	$-3.4 \cdot 10^{+00}$	$2.3 \cdot 10^{-06}$	4.736	$-3.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-3}$	$8.9 \cdot 10^{-08}$	4.806	$-4.4 \cdot 10^{+00}$	$8.9 \cdot 10^{-08}$	4.806	$-4.4 \cdot 10^{+00}$
$1/5 \cdot 2^{-4}$	$3.1 \cdot 10^{-09}$	4.883	$-6.2 \cdot 10^{+00}$	$3.1 \cdot 10^{-09}$	4.883	$-6.2 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$1.0 \cdot 10^{-10}$	4.936	$-8.2 \cdot 10^{+00}$	$1.0 \cdot 10^{-10}$	4.796	$-4.0 \cdot 10^{+00}$
$1/5 \cdot 2^{-6}$	$3.5 \cdot 10^{-12}$	4.967	$-9.8 \cdot 10^{+00}$	$3.8 \cdot 10^{-12}$	4.890	$-6.9 \cdot 10^{+00}$
$1/5 \cdot 2^{-7}$	$1.1 \cdot 10^{-13}$	4.983	$-1.0 \cdot 10^{+01}$	$1.3 \cdot 10^{-13}$	4.947	$-1.0 \cdot 10^{+01}$
$1/5 \cdot 2^{-8}$	$3.5 \cdot 10^{-15}$	4.991	$-1.1 \cdot 10^{+01}$	$4.2 \cdot 10^{-15}$	4.974	$-1.2 \cdot 10^{+01}$
$1/5 \cdot 2^{-9}$	$1.1 \cdot 10^{-16}$	4.995	$-1.1 \cdot 10^{+01}$	$1.3 \cdot 10^{-16}$	4.987	$-1.3 \cdot 10^{+01}$
$1/5 \cdot 2^{-10}$	$3.5 \cdot 10^{-18}$	4.997	$-1.2 \cdot 10^{+01}$	$4.2 \cdot 10^{-18}$	4.993	$-1.4 \cdot 10^{+01}$
$1/5 \cdot 2^{-11}$	$1.1 \cdot 10^{-19}$	4.998	$-1.2 \cdot 10^{+01}$	$1.3 \cdot 10^{-19}$	4.995	$-1.4 \cdot 10^{+01}$

## “Emden DE” – Multiple Shooting (2)

- $N$  is increased
- $h$  proportional to  $\frac{1}{N}$  such that
  - 1 step of IVP-solver inside  $[t_i, t_{i+1}]$
- expected order  $O(h^5)$  in regular case

$h$	$\ \varepsilon_s\ _\infty$	$p$	$c$	$\ \varepsilon_h\ _\infty$	$p$	$c$
$1/5 \cdot 2^{-1}$	$5.1 \cdot 10^{-05}$	4.445	$-1.4 \cdot 10^{+00}$	$5.5 \cdot 10^{-05}$	4.546	$-1.9 \cdot 10^{+00}$
$1/5 \cdot 2^{-2}$	$2.3 \cdot 10^{-06}$	2.847	$-1.2 \cdot 10^{-02}$	$2.3 \cdot 10^{-06}$	2.806	$-1.0 \cdot 10^{-02}$
$1/5 \cdot 2^{-3}$	$3.3 \cdot 10^{-07}$	1.652	$-1.4 \cdot 10^{-04}$	$3.4 \cdot 10^{-07}$	1.637	$-1.4 \cdot 10^{-04}$
$1/5 \cdot 2^{-4}$	$1.0 \cdot 10^{-07}$	1.974	$-6.0 \cdot 10^{-04}$	$1.0 \cdot 10^{-07}$	1.971	$-6.1 \cdot 10^{-04}$
$1/5 \cdot 2^{-5}$	$2.6 \cdot 10^{-08}$	2.023	$-7.7 \cdot 10^{-04}$	$2.7 \cdot 10^{-08}$	2.021	$-7.9 \cdot 10^{-04}$
$1/5 \cdot 2^{-6}$	$6.5 \cdot 10^{-09}$	2.023	$-7.7 \cdot 10^{-04}$	$6.8 \cdot 10^{-09}$	2.022	$-8.0 \cdot 10^{-04}$
$1/5 \cdot 2^{-7}$	$1.6 \cdot 10^{-09}$	2.015	$-7.3 \cdot 10^{-04}$	$1.6 \cdot 10^{-09}$	2.014	$-7.6 \cdot 10^{-04}$
$1/5 \cdot 2^{-8}$	$4.0 \cdot 10^{-10}$	2.008	$-6.9 \cdot 10^{-04}$	$4.1 \cdot 10^{-10}$	2.008	$-7.2 \cdot 10^{-04}$
$1/5 \cdot 2^{-9}$	$9.9 \cdot 10^{-11}$	2.004	$-6.7 \cdot 10^{-04}$	$1.0 \cdot 10^{-10}$	2.004	$-7.0 \cdot 10^{-04}$
$1/5 \cdot 2^{-10}$	$2.4 \cdot 10^{-11}$	2.000	$-6.5 \cdot 10^{-04}$	$2.5 \cdot 10^{-11}$	2.000	$-6.8 \cdot 10^{-04}$

## “Emden DE” – Multiple Shooting (3)

- $N$  is increased **but**  $t_1 := \frac{1}{8}$  fixed
- $h$  proportional to  $\frac{1}{N}$  such that
  - several steps of IVP-solver inside  $[0, 1/8]$
  - 1 step of IVP-solver inside  $[t_i, t_{i+1}]$
- expected order  $O(h^5)$  in regular case

$h$	$\ \varepsilon_s\ _\infty$	$p$	$c$	$\ \varepsilon_h\ _\infty$	$p$	$c$
$1/5 \cdot 2^{-3}$	$3.3 \cdot 10^{-07}$	3.311	$-6.7 \cdot 10^{-02}$	$3.4 \cdot 10^{-07}$	3.351	$-7.9 \cdot 10^{-02}$
$1/5 \cdot 2^{-4}$	$3.3 \cdot 10^{-08}$	4.106	$-2.1 \cdot 10^{+00}$	$3.3 \cdot 10^{-08}$	4.100	$-2.1 \cdot 10^{+00}$
$1/5 \cdot 2^{-5}$	$1.9 \cdot 10^{-09}$	4.519	$-1.7 \cdot 10^{+01}$	$1.9 \cdot 10^{-09}$	4.525	$-1.8 \cdot 10^{+01}$
$1/5 \cdot 2^{-6}$	$8.4 \cdot 10^{-11}$	4.756	$-6.9 \cdot 10^{+01}$	$8.4 \cdot 10^{-11}$	4.757	$-7.0 \cdot 10^{+01}$
$1/5 \cdot 2^{-7}$	$3.1 \cdot 10^{-12}$	4.886	$-1.6 \cdot 10^{+02}$	$3.1 \cdot 10^{-12}$	4.886	$-1.6 \cdot 10^{+02}$
$1/5 \cdot 2^{-8}$	$1.8 \cdot 10^{-13}$	4.984	$-7.0 \cdot 10^{+02}$	$1.7 \cdot 10^{-13}$	4.984	$-6.5 \cdot 10^{+02}$
$1/5 \cdot 2^{-9}$	$5.6 \cdot 10^{-15}$	4.991	$-6.6 \cdot 10^{+02}$	$5.5 \cdot 10^{-15}$	4.995	$-6.3 \cdot 10^{+02}$
$1/5 \cdot 2^{-10}$	$1.7 \cdot 10^{-16}$	4.996	$-6.3 \cdot 10^{+02}$	$1.7 \cdot 10^{-16}$	4.998	$-6.2 \cdot 10^{+02}$
$1/5 \cdot 2^{-11}$	$5.4 \cdot 10^{-18}$	4.999	$-6.2 \cdot 10^{+02}$	$5.4 \cdot 10^{-18}$	4.999	$-6.1 \cdot 10^{+02}$
$1/5 \cdot 2^{-12}$	$1.6 \cdot 10^{-19}$	4.999	$-6.2 \cdot 10^{+02}$	$1.6 \cdot 10^{-19}$	4.999	$-6.1 \cdot 10^{+02}$

## Conclusion

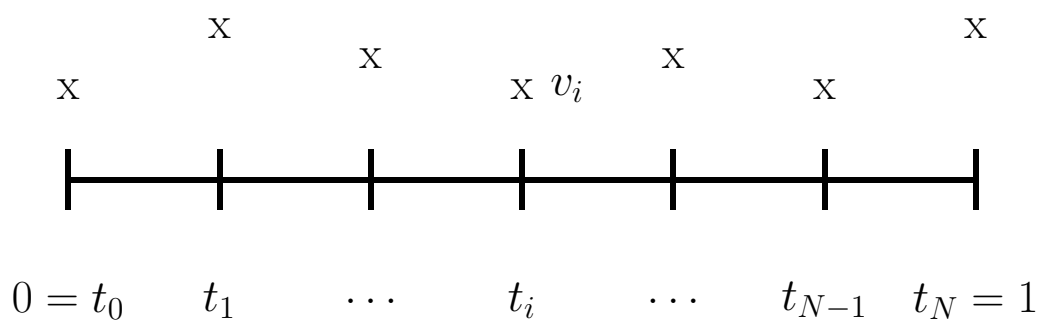
- IDeC is efficient
- Single Shooting is efficient  
if underlying IVP-integrator is efficient  
(Theory presented by O. Koch)
- Multiple Shooting works, but is not efficient

## Further Investigations

- integrate from right to left to allow  
 $\lambda(M(0)) : \operatorname{Re}(\lambda) > 0$  or  $\lambda = 0$
- idea how to couple both shooting methods  
to allow  $\lambda(M(0)) : \operatorname{Re}(\lambda) \neq 0$  or  $\lambda = 0$

## Implicit Euler method

- Runge-Kutta Methods lose their high convergence order.
- Remedy:
  - Choose a basic method of low order
  - Apply an acceleration technique



$$\Delta_h = (t_0, t_1, \dots, t_i, \dots, t_N), \quad N = 1/h$$

$$v_h = (v_0, v_1, \dots, v_i, \dots, v_N)$$

$\Downarrow$

$$|v_i - z(t_i)| = O(h), \quad i = 0(1)N, \quad \text{for } h \rightarrow 0$$

## Zadunaisky's idea

- solve

$$\begin{cases} z'(t) = F(t, z(t)), t \in (0, 1] \\ z(0) = \beta \end{cases}$$

by implicit Euler method

- obtain grid vector solution

$$z_h^{(0)} = (z_0^{(0)}, \dots, z_i^{(0)}, \dots, z_N^{(0)})$$

- interpolate  $(t_0, z_0^{(0)}), \dots, (t_N, z_N^{(0)})$  by polynomial  $p^{(0)}(t)$  of degree  $m = N$

- solve “neighboring problem”

$$\begin{cases} z'(t) = F(t, z(t)) + p^{(0)'}(t) - F(t, p^{(0)}(t)) \\ z(0) = p^{(0)}(0) = \beta \end{cases}$$

## Zadunaisky's idea (2)

- obtain grid vector solution  $p_h^{(0)}$  by implicit Euler method on the same grid  $\Delta_h$
- use known global error  $z_h^{(0)} - p_h^{(0)}$  as estimation for the unknown global error of the original problem
- accuracy by the heuristic argumentation
- implementation:
  - continuous function  $p^{(0)}(t)$  composed of polynomials with degree  $m < N$

## Iterated Defect Correction (IDeC)

- improve approximation

$$z_h^{(1)} := z_h^{(0)} + (z_h^{(0)} - p_h^{(0)})$$

- interpolate  $(t_0, z_0^{(1)}), \dots, (t_N, z_N^{(1)})$  by new polynomial  $p^{(1)}(t)$

- solve “neighboring problem” with defect

$$p^{(1)'}(t) - F(t, p^{(1)}(t))$$

- obtain grid vector solution  $p_h^{(1)}$  by implicit Euler method on the same grid  $\Delta_h$

- improve approximation again

$$z_h^{(2)} := z_h^{(0)} + (z_h^{(1)} - p_h^{(1)})$$