Analysis & Numerical Treatment of Singular IVPs

Part II

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Singular IVPs

$$\boxed{z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)),\ t \in (0,1]}$$

- $M(t) = M + tC(t)$, $C \in C[0,1]$
- $\lambda(M) : Re(\lambda) < 0$ or $\lambda = 0$
- $f(t,y) \in C([0,1] \times \mathbb{R}^n)$

$$\boxed{Mz(0) = 0} \qquad (\Longleftrightarrow z \in C[0,1])$$

- yields rank$(M) = n - m$ equations

$$\boxed{B_0 z(0) = \beta}$$

- $B_0 \in \mathbb{R}^{(m \times n)}$, $\beta \in \mathbb{R}^m$

Singular IVPs – theoretical results

- $C \in C^p[0,1]$, $p \geq 0$
- $f(t,y) \in C^p([0,1] \times \mathbb{R}^n)$,

  Lipschitz-condition with respect to y
- $m \times m$-matrix $B_0 R$ nonsingular,

  R projection onto eigenspace of M

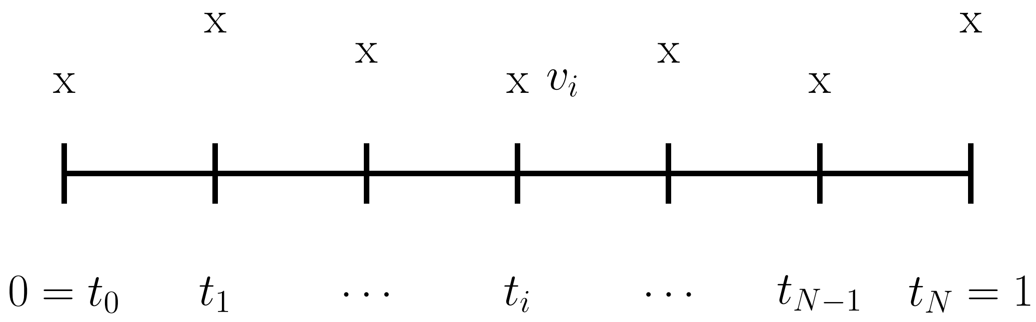
  for eigenvalue $\lambda = 0$

$$\Downarrow$$

existence and uniqueness of solution z

$z \in C^{p+1}[0,1]$ $(p = 6)$

Implicit Euler method



$$\Delta_h = (t_0, t_1, \ldots, t_i, \ldots, t_N),\ N = 1/h$$

$$\begin{cases} \frac{v_i - v_{i-1}}{h} = \frac{M(t_i)}{t_i} v_i + f(t_i, v_i),\ i = 1(1)N \\ B_0 v_0 = \beta \\ M v_0 = 0 \end{cases}$$

$$v_h = (v_0, v_1, \ldots, v_i, \ldots, v_N)$$

$$\Downarrow$$

$$|v_i - z(t_i)| = O(h),\ i = 0(1)N,\ \text{for}\, h \to 0$$

Asymtotic error expansion

- $v_h = (v_0, \ldots, v_i, \ldots, v_N)$ grid vector solution obtained by implicit Euler method on mesh Δ_h
- we make an ansatz

$$v_i = z(t_i) + \sum_{j=1}^{5} h^j e_j(t_i) + r_i,\ t_i \in \Delta_h$$

- and have to show
  - $\exists\, e_j$ smooth
  - $|r_i| = O(h^6)$, $i = 0(1)N$, for $h \to 0$

Variational equations

singular IVPs

$$\begin{cases} e_1'(t) - \frac{M(t)}{t} e_1(t) = f_y(t, z(t)) e_1(t) + \frac{1}{2} z''(t) \\ e_1(0) = 0 \end{cases}$$

$$\begin{cases} e_2'(t) - \frac{M(t)}{t} e_2(t) \ = \ f_y(t, z(t)) e_2(t) + \frac{1}{2} e_1''(t) + \\ \qquad\qquad\qquad\qquad \frac{1}{2} f_{yy}(t, z(t)) e_1^2(t) - \frac{1}{6} z^{(3)}(t) \\ e_2(0) = 0 \end{cases}$$

$$z \in C^7[0,1]$$

$$\Downarrow$$

$$e_1 \in C^6[0,1] \Longrightarrow e_2 \in C^5[0,1]$$

$$\Downarrow$$

e_j smooth for all further j, $3 \leq j \leq 5$

Remainder term

- r_i, $i = 1(1)N$, satisfies

$$\begin{cases} \frac{r_i - r_{i-1}}{h} = \frac{M(t_i)}{t_i} r_i + g(t_i, r_i) + l_i \\ r_0 = 0 \end{cases}$$

- with

$$g(t_i, r_i) := \int_0^1 f_y(t_i, v_i - (1-\tau) r_i)\, d\tau \cdot r_i$$

and $l_i = O(h^6)$

- by a contraction argument we show that there exists an $L < 1$ with

$$|r_i| \leq \text{const} \frac{1}{1-L} |l_i| = O(h^6), \text{ for } h \to 0$$

Zadunaisky's idea

- solve

$$\begin{cases} z'(t) = F(t, z(t)),\ t \in (0,1] \\ z(0) = \beta \end{cases}$$

by implicit Euler method

- obtain grid vector solution

$z_h^{(0)} = (z_0^{(0)}, \ldots, z_i^{(0)}, \ldots, z_N^{(0)})$

- interpolate $(t_0, z_0^{(0)}), \ldots, (t_N, z_N^{(0)})$ by polynomial $p^{(0)}(t)$ of degree $m = N$

- solve "neighboring problem"

$$\begin{cases} z'(t) = F(t, z(t)) + p^{(0)\prime}(t) - F(t, p^{(0)}(t)) \\ z(0) = p^{(0)}(0) = \beta \end{cases}$$

Zadunaisky's idea – continued

- obtain grid vector solution $p_h^{(0)}$ by implicit Euler method on the same grid Δ_h
- use known global error $z_h^{(0)} - p_h^{(0)}$ as estimation for the unknown global error of the original problem
- accuracy by the heuristic argumentation
- implementation:
  - continuous function $p^{(0)}(t)$ composed of polynomials with degree $m < N$

Iterated Defect Correction (IDeC)

- improve approximation

$$z_h^{(1)} := z_h^{(0)} + (z_h^{(0)} - p_h^{(0)})$$

- interpolate $(t_0, z_0^{(1)}), \ldots, (t_N, z_N^{(1)})$ by new polynomial $p^{(1)}(t)$
- solve "neighboring problem" with defect

$$p^{(1)\prime}(t) - F(t, p^{(1)}(t))$$

- obtain grid vector solution $p_h^{(1)}$ by implicit Euler method on the same grid Δ_h
- improve approximation again

$$z_h^{(2)} := z_h^{(0)} + (z_h^{(1)} - p_h^{(1)})$$

IDeC – expectations

$\surd$ practical basis for IDeC:

Zadunaisky's and Stetter's idea

$$n := \min(m, 6)$$

$$|z^{(j-1)}_{i^*_h} - z(t_*)| = O(h^j), \text{ for } 1 \leq j \leq n$$

$$|z^{(j-1)}_{i^*_h} - z(t_*)| = O(h^n), \text{ for } j > n$$

$\surd$ theoretical basis for IDeC:

asymtotic error expansion

"Emden differential equation"

$$\begin{cases} z'(t) = \frac{1}{t}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix} z(t) - t\begin{pmatrix}0\\ z_1^5(t)\end{pmatrix} \\ z(0) = (1,0)^T \end{cases}$$

h	$\|\varepsilon_h\|_h$	p	c	$\|\varepsilon_h^{(1)}\|_h$	$p^{(1)}$	$c^{(1)}$
$1/5$	$2.4{\cdot}10^{-02}$	0.834	$-9.2{\cdot}10^{-01}$	$7.7{\cdot}10^{-02}$	1.835	$-1.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-1}$	$1.3{\cdot}10^{-02}$	0.921	$-1.1{\cdot}10^{-01}$	$2.1{\cdot}10^{-03}$	1.918	$-1.7{\cdot}10^{-01}$
$1/5{\cdot}2^{-2}$	$7.1{\cdot}10^{-02}$	0.960	$-1.2{\cdot}10^{-01}$	$5.7{\cdot}10^{-03}$	1.959	$-2.0{\cdot}10^{-01}$
$1/5{\cdot}2^{-3}$	$3.6{\cdot}10^{-02}$	0.980	$-1.3{\cdot}10^{-01}$	$1.4{\cdot}10^{-04}$	1.979	$-2.1{\cdot}10^{-01}$
$1/5{\cdot}2^{-4}$	$1.8{\cdot}10^{-03}$	0.990	$-1.4{\cdot}10^{-01}$	$3.7{\cdot}10^{-04}$	1.989	$-2.2{\cdot}10^{-01}$
$1/5{\cdot}2^{-5}$	$9.3{\cdot}10^{-03}$	0.995	$-1.4{\cdot}10^{-01}$	$9.4{\cdot}10^{-05}$	1.994	$-2.3{\cdot}10^{-01}$
$1/5{\cdot}2^{-6}$	$4.7{\cdot}10^{-03}$	0.997	$-1.4{\cdot}10^{-01}$	$2.3{\cdot}10^{-06}$	1.997	$-2.3{\cdot}10^{-01}$
$1/5{\cdot}2^{-7}$	$2.3{\cdot}10^{-04}$	0.998	$-1.4{\cdot}10^{-01}$	$5.9{\cdot}10^{-06}$	1.998	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-8}$	$1.1{\cdot}10^{-04}$	0.999	$-1.5{\cdot}10^{-01}$	$1.4{\cdot}10^{-07}$	1.999	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-9}$	$5.8{\cdot}10^{-04}$	0.999	$-1.5{\cdot}10^{-01}$	$3.7{\cdot}10^{-07}$	1.999	$-2.4{\cdot}10^{-01}$
$1/5{\cdot}2^{-10}$	$2.9{\cdot}10^{-05}$	0.999	$-1.5{\cdot}10^{-01}$	$9.2{\cdot}10^{-08}$	2.000	$-2.4{\cdot}10^{-01}$

h	$\|\varepsilon_h^{(2)}\|_h$	$p^{(2)}$	$c^{(2)}$	$\|\varepsilon_h^{(3)}\|_h$	$p^{(3)}$	$c^{(3)}$
$1/5$	$1.6\cdot10^{-03}$	2.924	$-1.8\cdot10^{-01}$	$9.5\cdot10^{-03}$	3.330	$-2.0\cdot10^{-01}$
$1/5\cdot2^{-1}$	$2.1\cdot10^{-04}$	2.918	$-1.7\cdot10^{-01}$	$9.5\cdot10^{-04}$	3.525	$-3.1\cdot10^{+00}$
$1/5\cdot2^{-2}$	$2.8\cdot10^{-05}$	2.731	$-1.0\cdot10^{-01}$	$8.2\cdot10^{-05}$	3.770	$-6.6\cdot10^{+00}$
$1/5\cdot2^{-3}$	$4.3\cdot10^{-05}$	2.868	$-1.6\cdot10^{-01}$	$6.0\cdot10^{-06}$	3.889	$-1.0\cdot10^{+00}$
$1/5\cdot2^{-4}$	$5.9\cdot10^{-06}$	2.934	$-2.2\cdot10^{-01}$	$4.0\cdot10^{-07}$	3.946	$-1.3\cdot10^{+00}$
$1/5\cdot2^{-5}$	$7.7\cdot10^{-07}$	2.967	$-2.6\cdot10^{-01}$	$2.6\cdot10^{-09}$	3.973	$-1.5\cdot10^{+00}$
$1/5\cdot2^{-6}$	$9.8\cdot10^{-08}$	2.983	$-2.9\cdot10^{-01}$	$1.6\cdot10^{-10}$	3.974	$-1.5\cdot10^{+00}$
$1/5\cdot2^{-7}$	$1.2\cdot10^{-09}$	2.990	$-3.0\cdot10^{-01}$	$1.0\cdot10^{-11}$	3.613	$-1.4\cdot10^{-01}$
$1/5\cdot2^{-8}$	$1.5\cdot10^{-10}$	2.979	$-2.8\cdot10^{-01}$	$8.7\cdot10^{-12}$	0.958	$-8.3\cdot10^{-09}$

h	$\|\varepsilon_h^{(4)}\|_h$	$p^{(4)}$	$c^{(4)}$	$\|\varepsilon_h^{(5)}\|_h$	$p^{(5)}$	$c^{(5)}$
$1/5$	$8.1\cdot10^{-03}$	4.954	$-2.3\cdot10^{+00}$	$5.4\cdot10^{-03}$	5.558	$-4.1\cdot10^{+01}$
$1/5\cdot2^{-1}$	$2.6\cdot10^{-05}$	5.079	$-3.1\cdot10^{+00}$	$1.1\cdot10^{-05}$	4.921	$-9.5\cdot10^{+00}$
$1/5\cdot2^{-2}$	$7.7\cdot10^{-06}$	5.095	$-3.3\cdot10^{+01}$	$3.7\cdot10^{-06}$	5.141	$-1.8\cdot10^{+00}$
$1/5\cdot2^{-3}$	$2.2\cdot10^{-08}$	5.084	$-3.1\cdot10^{+01}$	$1.0\cdot10^{-08}$	5.102	$-1.6\cdot10^{+00}$
$1/5\cdot2^{-4}$	$6.6\cdot10^{-09}$	4.995	$-2.1\cdot10^{+00}$	$3.1\cdot10^{-10}$	5.080	$-1.4\cdot10^{+00}$
$1/5\cdot2^{-5}$	$2.0\cdot10^{-11}$	4.881	$-1.2\cdot10^{+00}$	$9.2\cdot10^{-11}$	5.395	$-7.2\cdot10^{+01}$
$1/5\cdot2^{-6}$	$7.1\cdot10^{-12}$	2.400	$-7.3\cdot10^{-06}$	$2.1\cdot10^{-13}$	0.770	$-1.8\cdot10^{-11}$

Example 2

$$\begin{cases} z'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(t) - t \begin{pmatrix} 0 \\ \mathrm{e}^{z_1(t)} - \frac{B^2-6B+1}{Bt^2+1} \end{pmatrix} \\ z(0) = (2\ln(B+1), 0)^T,\ B = 3 + \sqrt{8} \end{cases}$$

h	$\lVert\varepsilon_h\rVert_h$	p	c	$\lVert\varepsilon_h^{(1)}\rVert_h$	$p^{(1)}$	$c^{(1)}$
$1/5 \cdot 2^{-1}$	4.9	0.608	$-2.0 \cdot 10^{+00}$	$3.1 \cdot 10^{-01}$	1.385	$-7.5 \cdot 10^{+01}$
$1/5 \cdot 2^{-2}$	3.2	0.783	$-3.4 \cdot 10^{+01}$	$1.1 \cdot 10^{-01}$	1.682	$-1.8 \cdot 10^{+01}$
$1/5 \cdot 2^{-3}$	$1.8 \cdot 10^{-01}$	0.886	$-4.9 \cdot 10^{+01}$	$3.7 \cdot 10^{-01}$	1.838	$-3.2 \cdot 10^{+02}$
$1/5 \cdot 2^{-4}$	$1.0 \cdot 10^{-01}$	0.941	$-6.3 \cdot 10^{+01}$	$1.0 \cdot 10^{-02}$	1.917	$-4.6 \cdot 10^{+02}$
$1/5 \cdot 2^{-5}$	$5.3 \cdot 10^{-01}$	0.970	$-7.3 \cdot 10^{+01}$	$2.7 \cdot 10^{-03}$	1.958	$-5.6 \cdot 10^{+02}$
$1/5 \cdot 2^{-6}$	$2.7 \cdot 10^{-02}$	0.985	$-8.0 \cdot 10^{+01}$	$7.0 \cdot 10^{-03}$	1.979	$-6.4 \cdot 10^{+02}$
$1/5 \cdot 2^{-7}$	$1.3 \cdot 10^{-02}$	0.992	$-8.3 \cdot 10^{+01}$	$1.7 \cdot 10^{-04}$	1.989	$-6.8 \cdot 10^{+02}$
$1/5 \cdot 2^{-8}$	$6.9 \cdot 10^{-02}$	0.996	$-8.6 \cdot 10^{+01}$	$4.5 \cdot 10^{-04}$	1.994	$-7.1 \cdot 10^{+02}$
$1/5 \cdot 2^{-9}$	$3.4 \cdot 10^{-02}$	0.998	$-8.7 \cdot 10^{+01}$	$1.1 \cdot 10^{-05}$	1.997	$-7.2 \cdot 10^{+02}$
$1/5 \cdot 2^{-10}$	$1.7 \cdot 10^{-03}$	0.999	$-8.8 \cdot 10^{+01}$	$2.8 \cdot 10^{-06}$	1.998	$-7.3 \cdot 10^{+02}$

h	$\|\varepsilon_h^{(2)}\|_h$	$p^{(2)}$	$c^{(2)}$	$\|\varepsilon_h^{(3)}\|_h$	$p^{(3)}$	$c^{(3)}$
$1/5 \cdot 2^{-1}$	$1.7 \cdot 10^{-01}$	2.481	$-5.1 \cdot 10^{+02}$	$7.1 \cdot 10^{-01}$	3.125	$-9.5 \cdot 10^{+02}$
$1/5 \cdot 2^{-2}$	$3.0 \cdot 10^{-02}$	2.895	$-1.7 \cdot 10^{+02}$	$8.2 \cdot 10^{-02}$	3.107	$-9.0 \cdot 10^{+02}$
$1/5 \cdot 2^{-3}$	$4.1 \cdot 10^{-02}$	3.003	$-2.6 \cdot 10^{+02}$	$9.5 \cdot 10^{-03}$	3.524	$-4.2 \cdot 10^{+03}$
$1/5 \cdot 2^{-4}$	$5.1 \cdot 10^{-03}$	2.977	$-2.3 \cdot 10^{+02}$	$8.2 \cdot 10^{-04}$	3.806	$-1.4 \cdot 10^{+03}$
$1/5 \cdot 2^{-5}$	$6.5 \cdot 10^{-04}$	2.988	$-2.5 \cdot 10^{+02}$	$5.9 \cdot 10^{-05}$	3.915	$-2.5 \cdot 10^{+03}$
$1/5 \cdot 2^{-6}$	$8.2 \cdot 10^{-05}$	2.993	$-2.6 \cdot 10^{+02}$	$3.9 \cdot 10^{-06}$	3.960	$-3.2 \cdot 10^{+04}$
$1/5 \cdot 2^{-7}$	$1.0 \cdot 10^{-06}$	2.996	$-2.6 \cdot 10^{+02}$	$2.5 \cdot 10^{-08}$	3.980	$-3.7 \cdot 10^{+04}$
$1/5 \cdot 2^{-8}$	$1.2 \cdot 10^{-07}$	2.998	$-2.6 \cdot 10^{+02}$	$1.5 \cdot 10^{-09}$	3.979	$-3.7 \cdot 10^{+04}$
$1/5 \cdot 2^{-9}$	$1.6 \cdot 10^{-08}$	3.001	$-2.7 \cdot 10^{+02}$	$1.0 \cdot 10^{-10}$	3.666	$-3.1 \cdot 10^{+03}$

h	$\|\varepsilon_h^{(4)}\|_h$	$p^{(4)}$	$c^{(4)}$	$\|\varepsilon_h^{(5)}\|_h$	$p^{(5)}$	$c^{(5)}$
$1/5 \cdot 2^{-1}$	$3.0 \cdot 10^{-02}$	1.719	$-1.5 \cdot 10^{+00}$	$5.3 \cdot 10^{-01}$	2.829	$-3.6 \cdot 10^{+02}$
$1/5 \cdot 2^{-2}$	$9.2 \cdot 10^{-02}$	4.818	$-1.7 \cdot 10^{+04}$	$7.5 \cdot 10^{-02}$	6.257	$-1.0 \cdot 10^{+06}$
$1/5 \cdot 2^{-3}$	$3.2 \cdot 10^{-03}$	5.328	$-1.1 \cdot 10^{+05}$	$9.8 \cdot 10^{-04}$	4.749	$-3.9 \cdot 10^{+04}$
$1/5 \cdot 2^{-4}$	$8.1 \cdot 10^{-05}$	5.224	$-7.1 \cdot 10^{+05}$	$3.6 \cdot 10^{-05}$	5.272	$-3.9 \cdot 10^{+05}$
$1/5 \cdot 2^{-5}$	$2.1 \cdot 10^{-07}$	4.955	$-1.8 \cdot 10^{+04}$	$9.4 \cdot 10^{-07}$	5.315	$-4.9 \cdot 10^{+05}$
$1/5 \cdot 2^{-6}$	$7.0 \cdot 10^{-08}$	4.920	$-1.4 \cdot 10^{+04}$	$2.3 \cdot 10^{-09}$	5.256	$-3.4 \cdot 10^{+05}$
$1/5 \cdot 2^{-7}$	$2.3 \cdot 10^{-10}$	4.985	$-2.2 \cdot 10^{+04}$	$6.2 \cdot 10^{-10}$	5.552	$-2.3 \cdot 10^{+05}$

Conclusion

- order sequence is $O(h), O(h^2), O(h^3), \ldots$ as expected
- IDeC is very efficient

$$\Downarrow$$

further investigations:

- Is imlicit middle-point rule $O(h^2)$?
- Exists an asymtotic error expansion?
- Has the IDeC the order sequence $O(h^2)$, $O(h^4)$, $O(h^6), \ldots$?